DYNAMICS OF POLISH GROUPS

BY

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Abstract

This thesis consists of an introduction and two independent chapters.

In Chapter 2, we show that the group of all homeomorphisms of the Cantor set $H(2^{\mathbb{N}})$ has ample generics, that is, we show that for every m the diagonal conjugacy action $g \cdot (h_1, h_2, \ldots, h_m) = (gh_1g^{-1}, gh_2g^{-1}, \ldots, gh_mg^{-1})$ of $H(2^{\mathbb{N}})$ on $H(2^{\mathbb{N}})^m$ has a comeager orbit. This answers a question of Kechris and Rosendal. We prove that the generic tuple in $H(2^{\mathbb{N}})^m$ can be taken to be the limit of a certain projective Fraïssé family. We also give an example of a projective Fraïssé family, which has a simpler description than the one considered in the general case, and such that its limit is a homeomorphism of the Cantor set that has a comeager conjugacy class. These results will appear in [26]. Additionally, using the perspective of the projective Fraïssé theory, we give examples of measures on the Cantor set such that the generic measure preserving homeomorphism exists and is realized as a projective Fraïssé limit.

In Chapter 3, we prove that each measure preserving Boolean action by a Polish group of isometries of a locally compact separable metric space has a spatial model or, in other words, has a point realization. This result extends both a classical theorem of Mackey and a recent theorem of Glasner and Weiss, and it covers interesting new examples. In order to prove our result, we give a characterization of Polish groups of isometries of locally compact separable metric spaces which may be of independent interest. The solution to Hilbert's fifth problem plays an important role in establishing this characterization. This work is joint with Sławomir Solecki and is published in [28]. Additionally, using our characterization, we give an alternative proof of the result by Gao and Kechris saying that no continuous action by a Polish group of isometries of a locally compact separable metric space is turbulent.

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Chapter 1

Introduction

1.1 Polish groups

A *Polish group* is a separable and completely metrizable topological group. This class includes separable locally compact groups (in particular, Lie groups), but there are many more examples. An important class of Polish groups are *permutation groups*, that is, closed subgroups of the group of all permutations of the natural numbers equipped with the pointwise convergence metric. This class is equal to the class of automorphism groups of countable (model theoretic) structures. To study permutation groups we often use tools coming from model theory or Ramsey theory, see for example [25] or [39]. Permutation groups on the one hand share some properties with locally compact groups (they do not have whirly [14] or turbulent [20] actions), but, on the other hand, there are important phenomena that do not occur among locally compact groups and that do occur for some permutation groups (for example, extreme amenability [25]).

Polish groups come up in many areas of mathematics. Among important Polish groups are: the group of all unitary operators of the separable infinite-dimensional Hilbert space with the strong operator topology, the group of all measure preserving automorphisms of the standard Lebesgue space with the weak topology, groups of all isometries of Polish metric spaces (for example, of the Urysohn metric space), and groups of all homeomorphisms of compact metrizable spaces (for example: of the Cantor set, or of the pseudo-arc).

In Chapter 2, we study the group of all homeomorphisms of the Cantor set. This is an important example of a permutation group, studied in topological dynamics. In Chapter 3, we explore the structure and dynamics of Polish groups of isometries of locally compact separable metric spaces. This class includes locally compact separable groups, permutation groups, and many more (for instance, countable products of locally compact separable groups).

1.2 Ample generics

In this section we give an overview of results proved in Chapter 2.

A group G acts on itself by conjugation $g \cdot h = ghg^{-1}$. Orbits of this action are *conjugacy classes*. A classical result by Halmos asserts that $\mathrm{Aut}(X, \mu)$, the group of all measure preserving transformations of the Lebesgue space, has a dense conjugacy class; his proof uses the fundamental lemma due to Rokhlin. Motivated by this result, we say that a topological group has *RP (the Rokhlin property)* if it has a dense conjugacy class. It has *SRP (the strong Rokhlin property)* if it has a comeager ("large" in a topological sense) conjugacy class. A comeager conjugacy class necessarily has to be a G_δ (that is, an intersection of countably many open sets).

Hodges, Hodkinson, Lascar, and Shelah [18], and later Kechris and Rosendal [24], studied a much stronger notion of "largeness" of conjugacy classes. A topological group G has *m-ample generics* if it has SRP in dimension m, that is, if the diagonal conjugacy action of G on G^m:

$$g \cdot (h_1, h_2, \ldots, h_m) = (gh_1g^{-1}, gh_2g^{-1}, \ldots, gh_mg^{-1})$$

has a comeager orbit. It has *ample generics* if it has m-ample generics for every m.

We will call a tuple from this comeager orbit a *generic tuple*.

Groups with ample generics come up naturally in various contexts. Examples of such groups are:

- the group of all automorphisms of the random graph (Hrushovski [20]);
- the group of all isometries of the rational Urysohn space (Solecki [35]);
- the group of all Haar measure-preserving homeomorphisms of the Cantor set (Kechris-Rosendal [24]).

All known examples of groups with ample generics are permutation groups.

Polish groups with ample generics share many properties connecting their algebraic and topological structure. Kechris and Rosendal [24] showed that if G is a Polish group that has ample generics, then the conditions (1)-(3) below hold. See also [18] for earlier results.

(1) Every subgroup of G of index less than $2^{\aleph_0}$ is open (the small index property).

(2) The group G is not a union of countably many cosets of non-open subgroups.

(3) Every algebraic homomorphism from G to a separable topological group is continuous. (This condition implies that there is exactly one Polish group topology on G.)

A permutation group is *oligomorphic* if it has finitely many orbits on each $\mathbb{N}^n$. Equivalently, it is oligomorphic when it is an automorphism group of an $\aleph_0$-categorical structure. Kechris and Rosendal [24] showed that for an oligomorphic group G with ample generics the following condition holds.

(4) The group G has *the Bergman property*, that is, every action of G by isometries on a metric space has bounded orbits.

Denote the Cantor set by $2^{\mathbb{N}}$ and the group of homeomorphisms of the Cantor set by $H(2^{\mathbb{N}})$. Akin, Hurley, and Kennedy [4] and independently Glasner and Weiss [15] showed that $H(2^{\mathbb{N}})$ has the Rokhlin property. Later, this result was strengthened by Kechris and Rosendal [24] who showed that $H(2^{\mathbb{N}})$ has the strong Rokhlin property. Akin, Glasner, and Weiss [3] gave a different proof of this result. Moreover, they gave an explicit description of a generic homeomorphism of the Cantor set (that is, a homeomorphism with comeager conjugacy class).

The main result that is proved of Chapter 2 is the following.

Theorem 1.2.1. *The group of homeomorphisms of the Cantor set has ample generics.*

The main tool we use in the proof is the *projective Fraïssé theory* developed by Irwin and Solecki (see [21]). This is a dualization of the Fraïssé theory from model theory.

As $H(2^{\mathbb{N}})$ is (isomorphic to) an oligomorphic permutation group, as a consequence, we immediately get the following corollary.

Corollary 1.2.2. *The group of homeomorphisms of the Cantor set has properties (1)-(4).*

There is a common reason why all three above mentioned groups have ample generics. They all have the *Hrushovski property*. This property is absent in $H(2^{\mathbb{N}})$. We say that a structure X has the Hrushovski property if for every k, a finite substructure A of X, and a tuple of *partial* automorphisms $f_1, f_2, \ldots, f_k$ of A (a partial automorphism of A is an automorphism between two substructures of A), there exists a finite substructure $B \supseteq A$ of X together with automorphisms

$g_1, g_2, \ldots, g_k$ of B such that $g_i \upharpoonright A = f_i$. A permutation group G has the Hrushovski property if there exists a structure X with the Hrushovski property such that $G = \mathrm{Aut}(X)$.

It may be interesting to compare our results with the results by Hochman [17]. Let Γ be a countable discrete group. Let $\mathrm{Rep}(\Gamma, H(2^{\mathbb{N}}))$ be the set of all representations of Γ into $H(2^{\mathbb{N}})$ (we can also think of it as the set of all actions of Γ on $2^{\mathbb{N}}$ by homeomorphisms). This is a closed subset of $H(2^{\mathbb{N}})^{\Gamma}$. The group $H(2^{\mathbb{N}})$ acts on $\mathrm{Rep}(\Gamma, H(2^{\mathbb{N}}))$ by conjugation. When $\Gamma = F_m$, the free group on m generators, $\mathrm{Rep}(\Gamma, H(2^{\mathbb{N}}))$ can be identified with $H(2^{\mathbb{N}})^m$, and the action is the diagonal conjugacy action. Therefore, saying that $H(2^{\mathbb{N}})$ has m-ample generics is equivalent to saying that the action of $H(2^{\mathbb{N}})$ on $\mathrm{Rep}(F_m, H(2^{\mathbb{N}}))$ has a comeager orbit. In contrast, Hochman [17] showed that all orbits in the action of $H(2^{\mathbb{N}})$ ($m > 1$) on $\mathrm{Rep}(\mathbb{Z}^m, H(2^{\mathbb{N}}))$ are meager.

1.3 Groups of isometries

In this section we first discuss groups of isometries and then we give an overview of results proved in Chapter 3.

Given a metric separable space (X, d), by $\mathrm{Iso}(X)(= \mathrm{Iso}(X, d))$ we understand the group of all isometries of (X, d) with composition as group operation and with the topology of pointwise convergence. The group $\mathrm{Iso}(X)$ is a separable metrizable topological group. We say that a topological group G is a *group of isometries of* X if there exists an isomorphism that is also a homeomorphism between G and a subgroup of $\mathrm{Iso}(X)$.

Viewing Polish groups as isometry groups of metric spaces provides a natural stratification of the class of all Polish groups. The starting point here is the observation that if X is a Polish metric space, then $\mathrm{Iso}(X)$ is a Polish group. Furthermore, the following relevant results are known:

(i) (Uspenskij [40]) G is a Polish group of isometries of a Polish metric space if and only if G is a Polish group;

(ii) (Gao–Kechris [12, Theorem 6.3]) G is a Polish group of isometries of a locally compact Polish metric space, or equivalently, of a locally compact separable metric space, if and only if G is a closed subgroup of a group of the form

$$\prod_{n\in\mathbb{N}} S_\infty \ltimes M^{\mathbb{N}},$$

where S_∞ is the group of all permutations of $\mathbb{N}$, M is a locally compact second countable group, and the semidirect product is formed by viewing each $\sigma \in S_\infty$ as an automorphism of $M^{\mathbb{N}}$ which acts on $h \in M^{\mathbb{N}}$ by returning $\sigma(h) \in M^{\mathbb{N}}$ given by

$$\sigma(h)(i) = h(\sigma^{-1}(i));$$

(iii) (folklore) G is a Polish group of isometries of a proper metric space, that is, a metric space in which all balls are compact, if and only if G is a locally compact second countable group;

(iv) (folklore) G is a Polish group of isometries of a compact metric space if and only if G is a compact second countable group.

Moreover, in the conditions on the left hand side in each of the points above, one can replace the phrase “Polish group" by “closed group." Also, the conditions on the left hand side in each of the points above can be replaced by the condition that G be isomorphic to the whole group $\mathrm{Iso}(X)$ in (i) for a Polish metric space X [12, Theorem 3.1(i)], in (ii) for a locally compact Polish metric space X [12, Theorem 6.3], in (iii) for a proper metric space X [31, Theorem 2.1], and in (iv) for a compact metric space X [32], respectively.

Of primary interest to us will be Polish groups that are groups of isometries of locally compact separable metric spaces. Both locally compact Polish groups and permutation groups are Polish groups of isometries of locally compact separable metric spaces. This can be deduced from the above mentioned theorem due to Gao and Kechris [12] or, we can observe directly that a locally compact Polish group acts faithfully on itself by left translations preserving a left invariant metric, and a permutation group has a natural faithful action on $\mathbb{N}$ preserving the metric assigning distance 1 to each pair of distinct points in $\mathbb{N}$.

In a joint work with Sławomir Solecki we explore the following problem:

Question. For which topological groups G, does each Boolean action of G have a spatial model?

Let $(X, \mathcal{B}(X), \mu)$ be a standard Lebesgue space (i.e., there is a Polish topology on X whose family of Borel sets is $\mathcal{B}(X)$ and μ is a Borel probability measure on $\mathcal{B}(X)$). For $B \in \mathcal{B}(X)$, let $[B]_\mu$ be the μ-equivalence class of B. By $\mathcal{B}(X)/\mu$ we denote the Boolean algebra of all $[B]_\mu$, $B \in \mathcal{B}(X)$, with the usual Boolean operations. Let $\mathrm{Aut}(\mu)$ denote the Polish group of all measure preserving automorphisms of $(X, \mathcal{B}(X), \mu)$.

Let G be a Polish group. Assume we are given a continuous homomorphism $G \to \mathrm{Aut}(\mu)$, which we will view as a continuous action of G on $\mathcal{B}(X)/\mu$:

$$G \times \mathcal{B}(X)/\mu \ni (g, [B]_\mu) \to g \cdot [B]_\mu \in \mathcal{B}(X)/\mu.$$

We call such an action a (measure preserving) *Boolean action of* G *on* $\mathcal{B}(X)/\mu$. By a *spatial model* of such a Boolean action we mean a Borel action $G \times X \to X$ of G on X such that for each $B \in \mathcal{B}(X)$ and $g \in G$, we have

$$[g(B)]_\mu = g \cdot [B]_\mu.$$

In Section 3.1 we discuss two alternative, and equivalent, definitions of a Boolean action, and we precisely describe the topology on $\mathrm{Aut}(X, \mu)$.

By the classical theorem of Mackey [29] every Boolean action of a locally compact Polish group admits a spatial model. By the theorem of Glasner–Weiss [15, Theorem 2.3] every Boolean action of a permutation group admits a spatial model. There are also examples of Polish groups and their measure preserving Boolean actions without a spatial model. The natural Boolean action of $\mathrm{Aut}(\mu)$ on $\mathcal{B}(X)/\mu$ does not admit a spatial model. In Theorem 1.3.1, we extend the two results above to all Polish groups of isometries of locally compact separable metric spaces.

Theorem 1.3.1. *Let G be a Polish group of isometries of a locally compact separable metric space. Then each measure preserving Boolean action of G has a spatial model.*

The following characterization of Polish groups of isometries of locally compact separable metric spaces will be crucial in proving Theorem 1.3.1 and is of independent interest. Recall that if H is a subgroup of a group G, $N(H)$ stands for the normalizer of H, that is,

$$N(H) = \{g \in G : gHg^{-1} = H\}.$$

Theorem 1.3.2. *Let G be a Polish group. Then G is a group of isometries of a locally compact separable metric space if and only if each neighborhood of the identity contains a closed subgroup H such that the space G/H is locally compact and $N(H)$ is open.*

The proof of Theorem 1.3.2 uses the work of Gao and Kechris [12] on isometry groups of locally compact separable metric spaces, mentioned above. The main technical difficulty in our proof of Theorem 1.3.2 is showing that the property stated in that theorem is preserved under taking closed subgroups. Curiously, the argument establishing this preservation property uses the solution of Hilbert's fifth problem.

As a by-product of Theorem 1.3.2 we obtain the following corollary.

Corollary 1.3.3. *Let G be a Polish group of isometries of a locally compact separable metric space, and let N be a closed normal subgroup of G. Then G/N is also a Polish group of isometries of a locally compact separable metric space. In other words, the class of Polish groups of isometries of locally compact separable metric spaces is closed under taking images of continuous homomorphisms onto Polish groups.*

Chapter 2

Ample generics of the homeomorphism group of the Cantor set

2.1 Projective Fraïssé theory

We recall here basic notions and results on the projective Fraïssé theory developed by Irwin and Solecki in [21]. The projective Fraïssé theory is a dualization of the classical (injective) Fraïssé theory. Instead of countable (discrete) structures, we consider compact zero-dimensional second-countable structures. The Hereditary Property in the definition of the injective Fraïssé family is replaced by the condition (L2) in the definition of the projective Fraïssé limit.

Given a language L that consists of relation symbols $\{R_i\}_{i\in I}$, and function symbols $\{f_j\}_{\in J}$, a *topological L-structure* is a compact zero-dimensional second-countable space A equipped with closed relations R_i^A and continuous functions f_j^A, $i \in I, j \in J$. A continuous surjection $\phi\colon B \to A$ is an *epimorphism* if it preserves the structure, more precisely, for a function symbol f of arity n and $x_1, \ldots, x_n \in B$ we require:

$$f^A(\phi(x_1), \ldots, \phi(x_n)) = \phi(f^B(x_1, \ldots, x_n));$$

and for a relation symbol R of arity m and $x_1, \ldots, x_m \in B$ we require:

$$\begin{aligned} &(x_1, \ldots, x_m) \in R^A \\ &\iff \exists y_1, \ldots, y_m \in B \left(\phi(y_1) = x_1, \ldots, \phi(y_m) = x_m, \text{ and } (y_1, \ldots, y_m) \in R^B\right). \end{aligned}$$

By an *isomorphism* we mean a bijective epimorphism.

For the rest of this section fix a language L. Let $\mathcal{F}$ be a family of finite topological L-structures. We say that $\mathcal{F}$ is a *projective Fraïssé family* if the following two conditions hold:

(F1) (joint projection property: JPP) for any $A, B \in \mathcal{F}$ there are $C \in \mathcal{F}$ and epimorphisms from

C onto A and from C onto B;

(F2) (amalgamation property: AP) for $A, B_1, B_2 \in \mathcal{F}$ and any epimorphisms $\phi_1\colon B_1 \to A$ and $\phi_2\colon B_2 \to A$, there exist C, $\phi_3\colon C \to B_1$, and $\phi_4\colon C \to B_2$ such that $\phi_1 \circ \phi_3 = \phi_2 \circ \phi_4$.

A topological L-structure $\mathbb{L}$ is a *projective Fraïssé limit* of $\mathcal{F}$ if the following three conditions hold:

(L1) (projective universality) for any $A \in \mathcal{F}$ there is an epimorphism from $\mathbb{L}$ onto A;

(L2) for any finite discrete topological space X and any continuous function $f\colon \mathbb{L} \to X$ there are $A \in \mathcal{F}$, an epimorphism $\phi\colon \mathbb{L} \to A$, and a function $f_0\colon A \to X$ such that $f = f_0 \circ \phi$.

(L3) (projective ultrahomogeneity) for any $A \in \mathcal{F}$ and any epimorphisms $\phi_1\colon \mathbb{L} \to A$ and $\phi_2\colon \mathbb{L} \to A$ there exists an isomorphism $\psi\colon \mathbb{L} \to \mathbb{L}$ such that $\phi_2 = \phi_1 \circ \psi$;

Here is the fundamental result in the projective Fraïssé theory:

Theorem 2.1.1 (Irwin-Solecki, [21]). *Let $\mathcal{F}$ be a countable projective Fraïssé family of finite topological L-structures. Then:*

(1) there exists a projective Fraïssé limit of $\mathcal{F}$;

(2) any two topological L-structures that are projective Fraïssé limits are isomorphic.

In the propositions below we state some properties of the projective Fraïssé limit.

Proposition 2.1.2. *(1) If $\mathbb{L}$ is the projective Fraïssé limit the following condition (called* the extension property*) holds: Given $\phi_1\colon B \to A$, $A, B \in \mathcal{F}$, and $\phi_2\colon \mathbb{L} \to A$, then, there is $\psi\colon \mathbb{L} \to B$ such that $\phi_2 = \phi_1 \circ \psi$.*

(2) If $\mathbb{L}$ satisfies projective universality, the extension property, and (L2), then it also satisfies projective ultrahomogeneity, and therefore is isomorphic to the projective Fraïssé limit.

The projective Fraïssé limit is the inverse limit of certain topological L-structures from $\mathcal{F}$. More precisely, we have the following:

Proposition 2.1.3. *Let $\mathcal{F}$ be a countable projective Fraïssé family of finite topological L-structures. Let $\mathbb{L}$ be its projective Fraïssé limit. Then, there are $D_1, D_2, D_3, \ldots \in \mathcal{F}$ and $\pi_i\colon D_{i+1} \to D_i$ such that $\mathbb{L}$ is the inverse limit of*

$$D_1 \xleftarrow{\pi_1} D_2 \xleftarrow{\pi_2} D_3 \xleftarrow{\pi_3} \ldots,$$

and moreover, the following two properties hold:

(1) *For each $A \in \mathcal{F}$ there is i and there is an epimorphism $\phi\colon D_i \to A$.*

(2) *For all pairs of epimorphisms $\phi_1\colon B \to A$ and $\phi_2\colon D_i \to A$ there is $j > i$ and $\psi\colon D_j \to B$ such that $\phi_1 \circ \psi = \phi_2 \circ \pi_i^j$, where $\pi_i^j = \pi_i \circ \ldots \circ \pi_{j-1}$.*

For more background information on the projective Fraïssé theory and for proofs see [21] (the proof of Proposition 2.1.3 is included in the proof of Theorem 2.4 in [21], and the proof of Proposition 2.1.2 (ii) goes along the lines of the proof of the uniqueness of the projective Fraïssé limit in [21]). For a category-theoretic approach to related issues we refer the reader to [8].

Below we present several examples of projective Fraïssé families.

Example 2.1.4. Let $\mathcal{F}_0$ be the family of all finite sets (no structure). This is a projective Fraïssé family. The projective Fraïssé limit is the Cantor set.

Example 2.1.5 (Irwin-Solecki, [21])**.** Take the language L_1 that consists of one binary relation symbol R. Let $\mathcal{F}_1$ be the set of all finite reflexive linear graphs. We say that $A = (\{a_1, a_2, \ldots, a_n\}, R^A)$ is a finite reflexive linear graph if A is finite, and $R^A(x, y)$ if and only if $x = y$, or $x = a_i, y = a_{i+1}$ for some $i = 1, 2, \ldots, n-1$, or $x = a_{i+1}, y = a_i$ for some $i = 1, 2, \ldots, n-1$. It was shown in [21] that $\mathcal{F}_1$ is a projective Fraïssé family, its limit is a Cantor set equipped with a relation which is an equivalence relation with only one and two-element equivalence classes, and that the quotient of the limit is the pseudo-arc.

Example 2.1.6. Let $L = \{F\}$, where F is an unary function symbol. Consider

$$\mathcal{F}_2 = \{(A, F^A)\colon A \text{ is finite }, F^A \text{ is a bijection}\}.$$

This is a projective Fraïssé family. We check JPP and AP.

JPP: Take $(A, F^A), (B, F^B) \in \mathcal{F}_2$. Then $(A \times B, F^A \times F^B)$ together with projections works.

AP: Take $(A, F^A), (B, F^B), (C, F^C) \in \mathcal{F}_2$, $\phi_1\colon (B, F^B) \to (A, F^A)$, and $\phi_2\colon (C, F^C) \to (A, F^A)$. Then (D, F^D), where

$$D = \{(b, c) \in B \times C\colon \phi_1(b) = \phi_2(c)\}$$

and $F^D = F^B \times F^C$, together with projections works.

Denote the limit by $\mathbb{L} = (\mathbb{L}, F^{\mathbb{L}})$. The underlying set $\mathbb{L}$ is (homeomorphic to) the Cantor set. Since for every $(A, F^A) \in \mathcal{F}$, F^A is a bijection, it follows that $F^{\mathbb{L}}$ is a homeomorphism.

In fact, we can describe precisely the limit $(\mathbb{L}, F^{\mathbb{L}})$. The projective Fraïssé limit $(\mathbb{L}, F^{\mathbb{L}})$ is isomorphic to $(\Theta \times 2^{\mathbb{N}}, \tau \times \mathrm{id})$, where (Θ, τ) is the universal adding machine. The universal adding machine is the inverse limit of the inverse system $(\mathbb{Z}_{n!}, p_n^{n+1})_n$, where $\mathbb{Z}_{n!}$ is the ring of integers modulo $n!$, $p_n^{n+1}(k) = k \bmod n!$, and τ is the coordinatewise translation by the identity element. To show this, one can observe that $(\Theta \times 2^{\mathbb{N}}, \tau \times \mathrm{id})$ satisfies conditions (L1), (L2), and (L3) in the definition of the projective Fraïssé limit.

2.2 Spiral structures form a projective Fraïssé family

The goal of this section is to show that a generic homeomorphism of the Cantor set can be realized as a projective Fraïssé limit of the class of spiral structures (defined below). Many ideas in this section are motivated by [3].

Definition of a spiral structure. Let R be a binary relation symbol. We define a *spiral* $N = (N, R^N)$ to be the set $N = \{1, 2, \ldots, n\}$ with two distinguished points x_N and y_N such that $1 < x_N < y_N < n$ (we will be referring to them, respectively, as the *left node* of N and the *right node* of N), equipped with the relation R^N such that $R^N(i, i+1)$ for every $i = 1, 2, \ldots, n-1$, $R^N(x_N, 1)$, and $R^N(n, y_N)$. See also Figure 1.

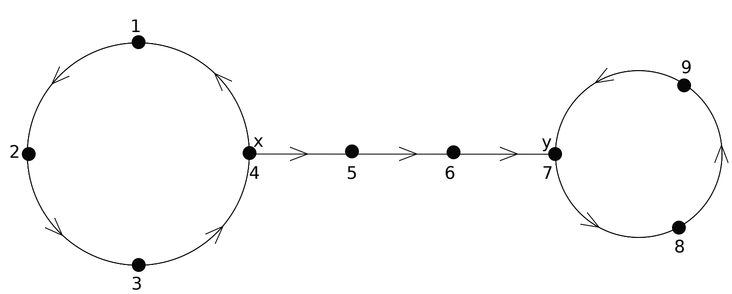

Figure 2.1: A spiral

We will call the interval $[1, x_N]$ the *left circle* of N and denote it by l_N, we will call the interval $[y_N, n]$ the *right circle* of N and denote it by r_N, and we will call the interval $[x_N, y_N]$ the *middle* *line* of N and denote it by s_N. Denote by $|l_N|$ the number of elements in the left circle in N, by $|s_N|$ the number of elements in the middle line of N, and by $|r_N|$ the number of elements in the right circle in N.

Spirals come up when we consider a homeomorphism of the Cantor set acting on clopen sets of the Cantor set. Take any $f \in H(2^{\mathbb{N}})$ and a clopen partition P of $2^{\mathbb{N}}$. For $p_0, p_1 \in P$ with $f(p_0) \cap p_1 \neq \emptyset$ we can choose (usually, in many ways) a bi-infinite sequence $(p_i)_{i \in \mathbb{Z}}$ with $f(p_i) \cap p_{i+1} \neq \emptyset$, $i \in \mathbb{Z}$, which is eventually periodic as $i \to +\infty$ and $i \to -\infty$; say $\ldots, p_{k-1}, p_k$ has period K, and $p_l, p_{l+1}, \ldots$ has period L, where $k < 0 < 1 < l$. Then, we can identify the sequence $p_{k-K+1}, \ldots, p_{k-1}, p_k, \ldots p_l, p_{l+1}, \ldots p_{l+L-1}$ with a spiral (p_l and p_k become the left and the right node, respectively). Notice that $f(p_i) \cap p_{i+1} \neq \emptyset$ for every $i = k-K+1, \ldots, l+L-2$, $f(p_l) \cap p_{l-L+1} \neq \emptyset$, and $f(p_{k+K-1}) \cap p_k \neq \emptyset$.

By a *spiral structure* we mean a disjoint union of spirals. Let $\mathcal{G}$ be the collection of all spiral structures. The main goal of this section is to show:

Theorem 2.2.1. *(1) The class $\mathcal{G}$ of spiral structures is a projective Fraïssé family.*

(2) The projective Fraïssé limit of $\mathcal{G}$ is a generic homeomorphism of the Cantor set.

Maps between spiral structures. We want to understand epimorphisms between two spiral structures. First note that:

Remark 2.2.2. Let $\phi\colon N \to M$ be an epimorphism between spiral structures. Then, the image of each spiral in N is contained in some spiral of M. Even more, it is either equal to a spiral in M, or it is equal to the left circle of a spiral in M, or it is equal to the right circle of a spiral in M.

It is therefore enough to describe only relation preserving maps (not necessarily surjective) between spirals. Before doing this precisely, let us see a *typical* example of a relation preserving map between spirals.

Example 2.2.3. Take $M = \{1, 2, 3, 4, 5, 6\}$ with $x_M = 3$ and $y_M = 5$. Take
$N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ with $x_N = 3$ and $y_N = 7$. The map $f\colon N \to M$ satisfying: $f(1) = 2$, $f(2) = 3$, $f(3) = 1$, $f(4) = 2$, $f(5) = 3$, $f(6) = 4$, $f(7) = 5$, $f(8) = 6$, $f(9) = 5$, and $f(10) = 6$ is relation preserving.

In the proposition below we collect information about relation preserving maps between spirals.

Proposition 2.2.4. *Let $M = \{1, 2, \dots, m\}$ and $N = \{1, 2, \dots, n\}$ be spirals. Let $f\colon N \to M$ be a relation preserving map. Let x be the left node of M and let y be the right node of M.*

(1) Suppose that f is onto M. Then, there are $a, b \in s_N$ such that $a < b$, $f(a) = x$, $f(b) = y$, and $b - a = |s_M|$ (there is exactly one such a pair (a, b)).

Conversely, suppose that $|s_M| \leq |s_N|$, $|l_M|$ divides $|l_N|$ and $|r_M|$ divides $|r_N|$. Given $a, b \in s_N$ such that $a < b$ and $b - a = |s_M|$, then, there is exactly one relation preserving $f\colon N \to M$ that is onto M, and such that $f(a) = x$ and $f(b) = y$.

(2) Given $f\colon N \to M$ that is onto the left circle of M, then, there is $c \in l_N$ such that $f(c) = x$ (there is more than one such c).

Conversely, suppose that $|l_M|$ divides $|l_N|$ and $|l_M|$ divides $|r_N|$. Given $c \in l_N$, then, there is exactly one relation preserving $f\colon N \to M$ that is onto the left circle of M and satisfies $f(c) = x$.

(3) Given $f\colon N \to M$ that is onto the right circle of M, then, there is $d \in r_N$ such that $f(d) = y$ (there is more than one such d).

Conversely, suppose that $|r_M|$ divides $|r_N|$ and $|r_M|$ divides $|l_N|$. Given $d \in r_N$, then, there is exactly one relation preserving $f\colon N \to M$ that is onto the right circle of M and satisfies $f(d) = y$.

Proof. In each of 1,2, and 3 the first statement is immediate, we just use that f is relation preserving.

For the second statement in 1, we define f in the following way: $f(b+k) = y+(k \bmod (m+1-y))$, for $k = 0, 1, \dots, n - b$; $f(a - k) = x - (k \bmod x)$, for $k = 0, 1, \dots, a - 1$; $f(k) = x + (k - a)$, for $a \leq k \leq b$. (Intuitively, everything to the left of a we wrap around the left circle of M, and everything to the right of b we wrap around the right circle of M.)

For the second statement in 2, we define f in the following way: $f(c + k) = k \bmod x$, for $k = 0, 1, \dots, n - c$ (here we identify 0 with x); $f(c - k) = x - (k \bmod x)$, for $k = 0, 1, \dots, c - 1$.

For the second statement in 3, we define f in the following way: $f(d+k) = y+(k \bmod (m+1-y))$, for $k = 0, 1, \dots, n - d$; $f(d - k) = (m + 1) - (k \bmod (m + 1 - y))$, for $k = 0, 1, \dots, d - 1$ (here we identify $m+1$ with y).

$\square$

Joint projection property. We check that $\mathcal{G}$ has the JPP. First take two spirals K and L. We want to find a spiral N that can be mapped both onto K and onto L. For this, let N be any spiral such that $|l_N|$ divides both $|l_L|$ and $|l_K|$, $|r_N|$ divides both $|r_L|$ and $|r_K|$, and $|s_N| > |s_K|, |s_L|$. We describe a relation preserving map from N onto K: Choose $a, b \in s_N$ with $a < b$ and $b - a = |s_K|$; map a to the left node of K, map b to the right node of K, and extend this to the map on the whole N. We similarly find a relation preserving map from N onto L.

In general, when K and L are spiral structures, for every pair of spirals in K and L we find a spiral that can be mapped onto both of them. The disjoint union of these spirals gives us the required spiral structure.

Amalgamation property. We check that $\mathcal{G}$ has the AP.

The general situation and strategy: We have a spiral structure $K_1 \cup \ldots \cup K_n$ (we have here a disjoint union of spirals), an epimorphism $\phi_1 \colon L_1 \cup \ldots \cup L_{n_1} \to K_1 \cup \ldots \cup K_n$, and an epimorphism $\phi_2 \colon M_1 \cup \ldots \cup M_{n_2} \to K_1 \cup \ldots \cup K_n$. Take L_i, and consider $\phi_1 \restriction L_i$. Its image is contained in some K_j. There are three possibilities: the image is equal to K_j, or it is equal to the left circle of K_j, or it is equal to the right circle of K_j.

For this fixed L_i, take any M_k such that $\phi_2 \restriction M_k$ is onto K_j. We find a spiral N, a relation preserving map $\phi_3 \colon N \to L_i$ that is onto, and a relation preserving map $\phi_4 \colon N \to M_k$ (we just want ϕ_4 to be into) such that $\phi_1 \circ \phi_3 = \phi_2 \circ \phi_4$. We do this with all of $L_1, L_2, \ldots, L_{n_1}$. Next, we proceed similarly with $M_1, M_2, \ldots, M_{n_2}$.

Therefore, it is enough to show the following:

Proposition 2.2.5. *Let K, L, M be spirals. Given a relation preserving map $f_1 \colon L \to K$ and a relation preserving map $f_2 \colon M \to K$ that is onto K, then, there exists a spiral N, a relation preserving map $f_3 \colon N \to L$ that is onto L, and a relation preserving map $f_4 \colon N \to M$ such that $f_1 \circ f_3 = f_2 \circ f_4$.*

Proof. Let x and y denote the left and right nodes of K, respectively. We consider the following three cases.

Case 1. The map f_1 is onto K. Here we will get f_4 that is onto M.

Take any spiral N such that $|l_N|$ divides both $|l_L|$ and $|l_M|$, $|r_N|$ divides both $|r_L|$ and $|r_M|$, and $|s_N| > 3(|s_M|+|s_L|)$. Take $a_1, b_1 \in s_L$ such that $a_1 < b_1$, $b_1 - a_1 = |s_K|$, $f_1(a_1) = x$, and $f_1(b_1) = y$. Take $a_2, b_2 \in s_M$ such that $a_2 < b_2$, $b_2 - a_2 = |s_K|$, $f_2(a_2) = x$, and $f_2(b_2) = y$. Choose $a, b \in s_N$ such that $a < b$ and $b - a = |s_K|$. Declare $f_3(a) = a_1$, $f_3(b) = b_1$, $f_4(a) = a_2$, $f_4(b) = b_2$. Extend f_3 and f_4 (in a unique way) to the whole N. We do this similarly as in the proof of Proposition 2.2.4 1. Above, we also have to make sure that our chosen a and b satisfy $a_1 - x_L, a_2 - x_M \leq a - x_N$ and $y_L - b_1, y_M - b_2 \leq y_N - b$.

Case 2. The map f_1 is onto l_K. Here we will get f_4 that is onto l_M.

Take any spiral N such that $|l_N|$ divides both $|l_L|$ and $|l_M|$, $|r_N|$ divides both $|r_L|$ and $|l_M|$, and $|s_N| > |l_L| + |s_L|$. Take $c_1 \in l_L$ such that $f_1(c_1) = x$. Take $c_2 \in l_M$ such that $f_2(c_2) = x$. Choose $c \in l_N$. Declare $f_3(c) = c_1$ and $f_4(c) = c_2$. Extend f_3 (in a non unique way) to the whole N so that f_3 is onto L. Extend f_4 (in a unique way) to the whole N so that f_4 is onto l_M.

Case 3. The map f_1 is onto r_K. Here we will get f_4 that is onto r_M.

Here we proceed as in Case 2. □

Let $(\mathbb{L}, R^{\mathbb{L}})$ denote the projective Fraïssé limit of $\mathcal{G}$.

Proposition 2.2.6. *The underlying set* $\mathbb{L}$ *is (homeomorphic to) the Cantor set.*

Proof. The underlying set $\mathbb{L}$ is compact, zero-dimensional, and second-countable, as $(\mathbb{L}, R^{\mathbb{L}})$ is a topological L-structure (where $L = \{R\}$). We show that $\mathbb{L}$ has no isolated points as follows. Suppose, towards a contradiction, that $p \in \mathbb{L}$ is an isolated point. Using (L2) find $A \in \mathcal{F}$ and an epimorphism $\phi\colon \mathbb{L} \to A$ such that the open cover $\{\{p\}, \mathbb{L} \setminus \{p\}\}$ is refined by $\{\phi^{-1}(a)\colon a \in A\}$. Set $a_0 = \phi(p)$. We can find B and $\bar{\phi}\colon B \to A$ such that there are distinct b_0, b_1 with $\bar{\phi}(b_0) = \bar{\phi}(b_1) = a_0$ (for example, take B equal to two disjoint copies of A, and require $\bar{\phi}$ restricted to each copy to be the identity). Using the extension property, find $\psi\colon \mathbb{L} \to B$ such that $\phi = \bar{\phi} \circ \psi$. Note that $\bar{\phi}^{-1}(b_0)$ and $\bar{\phi}^{-1}(b_1)$ are disjoint non-empty clopen subsets of $\{p\}$. This gives a contradiction. □

Proposition 2.2.7. *The closed relation* $R^{\mathbb{L}}$ *is the graph of a homeomorphism of the Cantor set.*

Proof. Suppose, towards a contradiction, that there are $\alpha, \beta_1, \beta_2 \in \mathbb{L}$, $\beta_1 \neq \beta_2$, such that $R^{\mathbb{L}}(\alpha, \beta_1)$ and $R^{\mathbb{L}}(\alpha, \beta_2)$. Take $A \in \mathcal{F}$ and $\psi_1\colon \mathbb{L} \to A$ such that $\psi_1(\beta_1) \neq \psi_1(\beta_2)$. Using the description of epimorphisms between spirals (Proposition 2.2.4) we observe that there is $B \in \mathcal{F}$ and $\phi\colon B \to A$ such that whenever x is such that $\phi(x) = \psi_1(\alpha)$, then there is exactly one $y \in B$ such that $R^B(x, y)$. Using the extension property find $\psi_2\colon \mathbb{L} \to B$ such that $\psi_1 = \phi \circ \psi_2$. We have $R^B(\psi_2(\alpha), \psi_2(\beta_1))$ and $R^B(\psi_2(\alpha), \psi_2(\beta_2))$. By the choice of ϕ, we get $\psi_2(\beta_1) = \psi_2(\beta_2)$, and therefore $\psi_1(\beta_1) = \psi_1(\beta_2)$. This gives a contradiction.

We similarly show that there are no $\alpha, \beta_1, \beta_2 \in \mathbb{L}$, $\beta_1 \neq \beta_2$, such that $R^{\mathbb{L}}(\beta_1, \alpha)$ and $R^{\mathbb{L}}(\beta_2, \alpha)$.

As $(\mathbb{L}, R^{\mathbb{L}})$ is a topological L-structure, $R^{\mathbb{L}}$ is closed and $\mathbb{L}$ is compact, and therefore, the function induced by $R^{\mathbb{L}}$, and its inverse, preserve the topology. □

Denote by $F^{\mathbb{L}}$ the function induced by $R^{\mathbb{L}}$. Below, we will be writing $(\mathbb{L}, F^{\mathbb{L}})$ rather than $(\mathbb{L}, R^{\mathbb{L}})$.

Proposition 2.2.8. *The conjugacy class of* $(\mathbb{L}, F^{\mathbb{L}})$ *is a dense* G_δ *in* $H(\mathbb{L}) = H(2^{\mathbb{N}})$.

Proof. The proof goes along the lines of proofs of Propositions 2.3.14 and 2.3.15, presented in the next section. □

2.3 $H(2^{\mathbb{N}})$ has ample generics

In this section we prove our main result saying that the homeomorphisms group of the Cantor set, $H(2^{\mathbb{N}})$, has ample generics.

Theorem 2.3.1. *The group of all homeomorphisms of the Cantor set has ample generics.*

First, we translate the question about the largeness of conjugacy classes in $H(2^{\mathbb{N}})^m$ into a combinatorial question about a family of finite directed graphs. More precisely, for a fixed m, the combinatorial question will concern a family $\mathcal{F}_0^m$ of finite structures, each equipped with m connected directed graphs with an extra surjectivity property, with structure preserving epimorphisms between structures in $\mathcal{F}_0^m$. Then, we show that there exists a subfamily $\mathcal{F}^m$ of $\mathcal{F}_0^m$ which satisfies the JPP (joint projection property) and the AP (amalgamation property), and is coinitial in $\mathcal{F}_0^m$. The properties JPP and AP make it possible to take a limit (the projective Fraïssé limit) of $\mathcal{F}^m$. Finally, we show that this limit can be viewed as a tuple in $H(2^{\mathbb{N}})^m$ and that its diagonal conjugacy class is comeager.

Let s be a symbol for a binary relation. Following [4] (Chapter 8) we say that s^A is a surjective relation on a set A if $s^A \subseteq A^2$ and for any $a \in A$ there are $b, c \in A$ such that $s^A(a,b)$ and $s^A(c,a)$. Note that s^A is a directed graph with an additional surjectivity property.

Surjective relations come up naturally as restrictions of homeomorphisms of the Cantor set to clopen partitions of the Cantor set. If P is a clopen partitions of $2^{\mathbb{N}}$ and $f \in H(2^{\mathbb{N}})$, then $\{(p,q) \in P^2 \colon f(p) \cap q \neq \emptyset\}$ is a surjective relation. We can think of a surjective relation as a partial homeomorphism of the Cantor set. Note also that spiral structures considered in the previous section are surjective relations.

To get a generic m-tuple of homeomorphisms, we will consider a certain family $\mathcal{F}$ of m-tuples of surjective relations (Theorem 2.3.2). After taking the limit, we obtain an m-tuple of closed relations on the Cantor set, which are surjective (that is, projections on both coordinates are onto). We show that every relation in this tuple is necessarily a permutation (Proposition 2.3.11), and therefore, is a graph of a homeomorphism of the Cantor set. Finally, we show that this m-tuple of homeomorphisms is generic.

Let $L = \{s_1, s_2, \ldots, s_m\}$, where $s_1, s_2, \ldots, s_m$ are symbols for binary relations. Let

$$\mathcal{F}_0 = \{(A, s_1^A, \ldots, s_m^A) \colon A \text{ is a finite non-empty set}, s_1^A, \ldots, s_m^A \text{ are surjective relations } \}.$$

It is straightforward to show that $\mathcal{F}_0$ has the JPP. Take $(A, s_1^A, \ldots, s_m^A)$, $(B, s_1^B, \ldots, s_m^B) \in \mathcal{F}_0$. Then $(A \times B, s_1^A \times s_1^B, \ldots, s_m^A \times s_m^B)$ together with projections as epimorphisms works.

We want to find a *coinitial* subfamily $\mathcal{F}$ of $\mathcal{F}_0$ (that is, such that for every $A \in \mathcal{F}_0$ there is $B \in \mathcal{F}$ and an epimorphism $\phi \colon B \to A$), which is a projective Fraïssé family. From the coinitiality of $\mathcal{F}_0$ it will follow that $\mathcal{F}$ has the JPP as well. The main difficulty is to take care of the AP.

We start with some notation. Let $s_1^{-1}, s_2^{-1}, \ldots, s_m^{-1}$ be symbols for the inverses of $s_1, s_2, \ldots, s_m$. For R equal to $s_1, s_1^{-1}, \ldots, s_m, s_m^{-1}$, R^{-1} denotes $s_1^{-1}, s_1, \ldots, s_m^{-1}, s_m$, respectively. Given $A = (A, s_1^A, \ldots, s_m^A)$, then $(s_1^{-1})^A, \ldots, (s_m^{-1})^A$ are surjective relations too. Let R be one of $s_1, s_1^{-1}, \ldots, s_m, s_m^{-1}$. Given $x \in A$, we say that x is R^A-*outgoing* if there is more than one $z \in A$ with $R^A(x,z)$, and there is exactly one $y \in A$ with $R^A(y,x)$. We say that x is R^A-*incoming* if there is more than one $y \in A$ with $R^A(y,x)$, and there is exactly one $z \in A$ with $R^A(x,z)$. Note that x is R^A-outgoing iff it is $(R^{-1})^A$-incoming.

For $A \in \mathcal{F}_0$ we say that we can *amalgamate over* A if for any $B, C \in \mathcal{F}_0$, $\phi_1 \colon B \to A$, and $\phi_2 \colon C \to A$ there are $D \in \mathcal{F}_0$, $\phi_3 \colon D \to B$, and $\phi_4 \colon D \to C$ such that $\phi_1 \circ \phi_3 = \phi_2 \circ \phi_4$.

Let $\mathcal{F}$ be the collection of all structures from $\mathcal{F}_0$ that satisfy (i) and (ii) of Theorem 2.3.2 below. From the coinitiality of $\mathcal{F}$ in $\mathcal{F}_0$ (Theorem 2.3.7 below) and Theorem 2.3.2 it will follow that $\mathcal{F}$ is a projective Fraïssé family.

Theorem 2.3.2. *Given $A = (A, s_1^A, \dots, s_m^A)$, suppose that A satisfies the following conditions.*

(1) Every point in A is outgoing for exactly one of $s_1^A, (s_1^{-1})^A, \dots, s_m^A, (s_m^{-1})^A$.

(2) Let R be one of $s_1, s_2, \dots, s_m$. Suppose that $R^A(x, y)$. Then either x is R^A-outgoing or y is R^A-incoming.

Then we can amalgamate over A.

Remark 2.3.3. Condition 2 of Theorem 2.3.2 implies that if R is one of $s_1^{-1}, s_2^{-1}, \dots, s_m^{-1}$ and if $R^A(x, y)$, then either x is R^A-outgoing or y is R^A-incoming.

Proof of Theorem 2.3.2. Given $A = (A, s_1^A, \dots, s_m^A), B = (B, s_1^B, \dots, s_m^B), C = (C, s_1^C, \dots, s_m^C)$, $\phi_1 \colon B \to A$, $\phi_2 \colon C \to A$, we want to find D, $\phi_3 \colon D \to B$ and $\phi_4 \colon D \to C$ such that $\phi_1 \circ \phi_3 = \phi_2 \circ \phi_4$.

We start with some definitions. We let

$$D_0 = \{(b, c) \in B \times C \colon \phi_1(b) = \phi_2(c)\}.$$

For R equal to one of $s_1, s_1^{-1}, \dots, s_m, s_m^{-1}$ we let

$$R^{D_0} = \{((b, c), (b', c')) \in D_0 \times D_0 \colon (b, b') \in R^B, (c, c') \in R^C\}.$$

Let $\pi_1 \colon D_0 \to B$ and $\pi_2 \colon D_0 \to C$ be the projections. (We will also write π_1, π_2 for restrictions of π_1, π_2 to subsets of D_0.) The surjectivity of π_1 and π_2 follows from the surjectivity of ϕ_1 and ϕ_2.

The relations $s_1^{D_0}, \dots, s_m^{D_0}$ do not have to be surjective. We find $D \subseteq D_0$ so that $s_1^D = s_1^{D_0} \restriction D, \dots, s_m^D = s_2^{D_0} \restriction D$ are surjective. For $n = 1, 2, 3, \dots$ we let

$$\begin{aligned} D_n = \{&(x', x'') \in D_{n-1} \colon \text{ for every } R = s_1, s_1^{-1}, \dots, s_m, s_m^{-1} \text{ there is} \\ &(y', y'') \in D_{n-1} \text{ such that } R^{D_0}((x', x''), (y', y''))\}. \end{aligned}$$

Let $D = \bigcap_n D_n$. Clearly $s_1^{D_0} \restriction D, \ldots, s_m^{D_0} \restriction D$ are surjective. We show that $\pi_1 \colon D \to B$ and $\pi_2 \colon D \to C$ are epimorphisms (Lemma 2.3.6).

Define $E_0 = D_0$. Let $x \in A$. Let R be such that x is R^A-outgoing. For $n = 1, 2, 3, \ldots$ we define

$$\begin{aligned} E_n^x = &\{(x', x'') \in E_{n-1} \colon x = \phi_1(x') = \phi_2(x'') \text{ and there is } (y', y'') \in E_{n-1} \\ &\text{such that } R^{D_0}((x', x''), (y', y''))\}, \end{aligned}$$

and let $E_n = \bigcup_{x \in A} E_n^x$.

Lemma 2.3.4. *Let $x \in A$. Let R be such that x is R^A-outgoing. Let n be a positive natural number. Suppose that $(x', x'') \in E_0$ with $\phi_1(x') = \phi_2(x'') = x$, $(y', y'') \in E_{n-1}$, and $R^{D_0}((x', x''), (y', y''))$. Then $(x', x'') \in E_n$.*

Proof. We have $(x', x'') \in E_0$ and $(y', y'') \in E_i$, for every $i = 0, 1, \ldots, n-1$. Furthermore, for every natural number j, if $(x', x'') \in E_j$ and $(y', y'') \in E_j$, then $(x', x'') \in E_{j+1}$. This gives us $(x', x'') \in E_n$. □

Lemma 2.3.5. *We have $E_n = D_n$ for every n.*

Proof. This is clear for $n = 0$. Suppose it holds for n, and we prove it for $n+1$. Clearly $D_{n+1} \subseteq E_{n+1}$. We show $E_{n+1} \subseteq D_{n+1}$. Take $(x', x'') \in E_{n+1}$. So $(x', x'') \in E_n = D_n$.

First let R be such that $x = \phi_1(x') = \phi_2(x'')$ is R^A-outgoing. Then, from the definition of E_{n+1}^x, there is $(y', y'') \in E_n = D_n$ such that $R^{D_0}((x', x''), (y', y''))$.

Now let R be such that x is not R^A-outgoing. Take $y \in A$ such that $R^A(x, y)$. Since x is not R^A-outgoing, y is R^A-incoming. Take any $y' \in B$ and $y'' \in C$ such that $R^B(x', y')$ and $R^C(x'', y'')$. Again, since x is not R^A-outgoing, $y = \phi_1(y') = \phi_2(y'')$, so $(y', y'') \in E_0$. From the fact that y is $(R^{-1})^A$-outgoing and $(R^{-1})^{D_0}((y', y''), (x', x''))$, by Lemma 2.3.4, we get $(y', y'') \in E_{n+2}$. Therefore $(y', y'') \in E_n = D_n$. We have proved that $(x', x'') \in D_{n+1}$. □

Lemma 2.3.6. *For every $n = 0, 1, 2, \ldots$:*

$(i)_n$ *$\pi_1[E_n] = B$;*

$(ii)_n$ *for $x', y' \in B$ with $R^B(x', y')$, where R is one of $s_1, s_1^{-1}, \ldots, s_m, s_m^{-1}$, there are $x'', y'' \in C$ such that $R^C(x'', y'')$, $\phi_1(x') = \phi_2(x'')$, $\phi_1(y') = \phi_2(y'')$, and $(x', x''), (y', y'') \in E_n$.*

of Lemma 2.3.6. **Proof of** $(i)_0$**:** Clear.

Proof that $(ii)_n$ **implies** $(i)_{n+1}$: Let $x' \in B$ be given. Let R be such that $x = \phi_1(x')$ is R^A-outgoing. Take any $y' \in B$ such that $R^B(x', y')$. Now from $(ii)_n$ we get $x'', y'' \in C$ such that $(x', x''), (y', y'') \in E_n$ and $R^C(x'', y'')$. From the definition of E^x_{n+1} we get $(x', x'') \in E_{n+1}$.

Proof that $(i)_n$ **implies** $(ii)_n$: Let R and $x', y' \in B$ with $R^B(x', y')$ be given. Let $x = \phi_1(x') = \phi_2(y')$. We can assume that x is R^A-outgoing. (Otherwise, y is $(R^{-1})^A$-outgoing and the proof is the same.)

If y is R^A-incoming, then take any $x'', y'' \in C$ with $\phi_2(x'') = x$, $\phi_2(y'') = y$, and $R^C(x'', y'')$. So $R^{D_0}((x', x''), (y', y''))$. Since x is R^A-outgoing and y is $(R^{-1})^A$-outgoing, from Lemma 2.3.4 we get $(x', x''), (y', y'') \in E_n$.

If y is not R^A-incoming, use $(i)_n$ to find $y'' \in C$ such that $(y', y'') \in E_n$. Now take any $x'' \in C$ such that $R^C(x'', y'')$. Then since y is not R^A-incoming, we have $\phi_2(x'') = x$. Note further that since $R^{D_0}((x', x''), (y', y''))$, from the definition of E^x_{n+1}, we get $(x', x'') \in E_{n+1} \subseteq E_n$. This shows $(ii)_n$. □

Since there clearly is n such that $D = E_n$, Lemma 2.3.6 implies that π_1 is an epimorphism. We similarly show that π_2 is an epimorphism. Therefore $\phi_3 = \pi_1 \restriction D$ and $\phi_4 = \pi_2 \restriction D$ work.

□

Theorem 2.3.7. *The collection of all $B = (B, s^B_1, \dots, s^B_m)$ satisfying the hypotheses of Theorem 2.3.2 is coinitial in $\mathcal{F}$.*

Proof. Given $A = (A, s^A_1, \dots, s^A_m)$, we take $4m$ disjoint copies of A. Call them $A^{+s_i}, \widehat{A}^{+s_i}, A^{-s_i}, \widehat{A}^{-s_i}$, $i = 1, 2, \dots, m$. Now we define $B = (B, s^B_1, \dots, s^B_m)$. Let

$$B = \bigcup_i \left(A^{+s_i} \cup \widehat{A}^{+s_i} \cup A^{-s_i} \cup \widehat{A}^{-s_i} \right)$$

be the underlying set.

First some notation. Let R be one of $s_1, \dots, s_m$. For $a \in A$, the copy of a in A^{+R} will be denoted by $a(A^{+R})$, etc. For $b \in B$, by $p(b)$ we denote the corresponding element in A.

Now we define R^B.

(1) For every $(x, y) \in R^A$ we put $(x(A^{+R}), y(A^{-R}))$, $(x(\widehat{A}^{+R}), y(A^{-R}))$, $(x(A^{+R}), y(\widehat{A}^{-R}))$, and $(x(\widehat{A}^{+R}), y(\widehat{A}^{-R}))$ into R^B.

(2) For every $b \in B$ choose exactly one $a \in A$ such that $(a, p(b)) \in R^A$, and put $(a(A^{+R}), b)$ into R^B.

(3) For every $b \in B$ choose exactly one $a' \in A$ such that $(p(b), a') \in R^A$, and put $(b, a'(A^{-R}))$ into R^B.

The relations s_1^B and s_2^B are surjective and the natural projection from B onto A is an epimorphism. We show that $(B, s_1^B, \ldots, s_m^B)$ is as needed.

Claim 2.3.8. *The structure B satisfies the hypotheses of Theorem 2.3.2.*

Proof. Let $i = 1, 2, \ldots, m$. From the definition of s_i^B, the s_i^B-outgoing points are exactly $a(A^{+s_i})$ and $a(\widehat{A}^{+s_i})$, $a \in A$, and $(s_i^{-1})^B$-outgoing points are exactly $a(A^{-s_i})$ and $a(\widehat{A}^{-s_i})$, $a \in A$. From this we get 1 of Theorem 2.3.2. From (i), (ii) and (iii) in the definition of R^B it is clear that 2 of Theorem 2.3.2 is also satisfied.

□

□

In the rest of this section we show:

Theorem 2.3.9. *The projective Fraïssé limit of $\mathcal{F}$ is a generic tuple in $H(2^{\mathbb{N}})^m$.*

Denote the projective Fraïssé limit of $\mathcal{F}$ by $\mathbb{L} = (\mathbb{L}, s_1^{\mathbb{L}}, \ldots, s_m^{\mathbb{L}})$. First we show that closed relations $s_1^{\mathbb{L}}, \ldots, s_m^{\mathbb{L}}$ are graphs of homeomorphisms of the Cantor set, and then we show that the homeomorphisms induced by $s_1^{\mathbb{L}}, \ldots, s_m^{\mathbb{L}}$ form a generic tuple, that is, the diagonal conjugacy class of this tuple is comeager. We borrow some ideas from [3] (from the proofs of Proposition 3.2 and Theorem 3.3 in [3]).

Let

$$\mathcal{G}_0 = \{(A, s^A) \colon A \text{ is a finite set and } s^A \text{ is a surjective relation}\}.$$

Lemma 2.3.10. *The family $\mathcal{G}$ of spiral structures (defined in Section 2) is coinitial in $\mathcal{G}_0$.*

Proof. Take any $A \in \mathcal{G}_0$. Take $x_0, x_1 \in A$ with $R^A(x_0, x_1)$. Note that the pair (x_0, x_1) can be extended to a bi-infinite sequence $(x_i)_{i\in\mathbb{Z}}$ with $R^A(x_i, x_{i+1})$, $i \in \mathbb{Z}$, which is eventually periodic as $i \to +\infty$ and $i \to -\infty$. From this we get a spiral $M = M_{(x_0,x_1)}$ and a relation preserving map $f\colon M \to A$ such that for some $x_0', x_1' \in M$ with $R^M(x_0', x_1')$, $f(x_0') = x_0$ and $f(x_1') = x_1$. The required spiral structure is the disjoint union

$$\bigcup_{\{(x_0,x_1)\in A^2\colon R^A(x_0,x_1)\}} M_{(x_0,x_1)}.$$

□

Proposition 2.3.11. *The closed relations $s_1^{\mathbb{L}}, \ldots, s_m^{\mathbb{L}}$ are graphs of homeomorphisms of the Cantor set.*

Proof. In Proposition 2.2.7 we showed that the projective Fraïssé limit of $\mathcal{G}_0$ is the graph of a homeomorphism of the Cantor set. In Lemma 2.3.10 we showed that $\mathcal{G}$ is coinitial in $\mathcal{G}_0$. Let

$$\mathcal{G}' = \{(A, s_1^A)\colon \text{ there are } s_2^A, \ldots, s_m^A \text{ such that } (A, s_1^A, \ldots, s_m^A) \in \mathcal{F}\}.$$

This also is a coinitial in $\mathcal{G}_0$ projective Fraïssé family.

The projective Fraïssé limits of $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic to each other, and they are also isomorphic to $(\mathbb{L}, s_1^{\mathbb{L}}), (\mathbb{L}, s_2^{\mathbb{L}}), \ldots, (\mathbb{L}, s_m^{\mathbb{L}})$. In particular, $s_1^{\mathbb{L}}, s_2^{\mathbb{L}}, \ldots, s_m^{\mathbb{L}}$ are graphs of homeomorphisms of the Cantor set $\mathbb{L}$. □

Remark 2.3.12. One can give a more direct, not referring to Section 3, proof of Proposition 2.3.11. For example, one can adapt the proof of Proposition 2.2.7 to our situation.

We denote the homeomorphisms whose graphs are $s_1^{\mathbb{L}}, \ldots, s_m^{\mathbb{L}}$ by $F_1^{\mathbb{L}}, \ldots, F_m^{\mathbb{L}}$, respectively. We also write $(\mathbb{L}, F_1^{\mathbb{L}}, \ldots, F_m^{\mathbb{L}})$ rather than $(\mathbb{L}, s_1^{\mathbb{L}}, \ldots, s_m^{\mathbb{L}})$.

By P or Q we denote partitions of $2^{\mathbb{N}}$. All partitions will be clopen partitions. For $f \in H(2^{\mathbb{N}})$ and a partition P we define

$$f \restriction P = \{(p, q) \in P^2\colon f(p) \cap q \neq \emptyset\}.$$

This is a surjective relation. Let $(f_1,\dots,f_m)\restriction P=(f_1\restriction P,\dots,f_m\restriction P)$. Define

$$[P,s_1^P,\dots,s_m^P]=\{(f_1,\dots,f_m)\in H(2^{\mathbb{N}})^m\colon f_1\restriction P=s_1^P,\dots,f_m\restriction P=s_m^P\}.$$

Lemma 2.3.13. *Sets of the form $[P,s_1^P,\dots,s_m^P]$ are clopen in $H(2^{\mathbb{N}})^m$. Moreover, they form a topological basis in $H(2^{\mathbb{N}})^m$.*

Proof. Clearly they are clopen sets. Take any $(g_1,\dots,g_m)\in H(2^{\mathbb{N}})^m$, and take $\epsilon>0$. Let $U=\{(f_1,\dots,f_m)\colon \forall i\forall x\ d(f_i(x),g_i(x))<\epsilon\}$ (here d is any metric on $2^{\mathbb{N}}$). This is an open set. We want to find a clopen neighborhood of $(g_1,\dots,g_m)$ that is of the form $[P,s_1^P,\dots,s_m^P]$ and is contained in U. For this, take first an arbitrary partition Q of $2^{\mathbb{N}}$ of mesh $<\epsilon$, and $P=\{q_0\cap g_1^{-1}(q_1)\cap\ldots\cap g_m^{-1}(q_m)\colon q_0,q_1,\dots,q_m\in Q\}$. For $i=1,2,\dots,m$, we let $s_i^P=\{(p,r)\colon g_i(p)\cap r\neq\emptyset\}$. Clearly $(g_1,\dots,g_m)\in[P,s_1^P,\dots,s_m^P]$. Now take any $(f_1,\dots,f_m)\in[P,s_1^P,\dots,s_m^P]$, and $p\in P$, say $p=q_0\cap g_1^{-1}(q_1)\cap\ldots\cap g_m^{-1}(q_m)$. Then $g_i(p)\subseteq q_i$ for every $i=1,2,\dots,m$. For any $r\in P$, $f_i(p)\cap r\neq\emptyset$ iff $g_i(p)\cap r\neq\emptyset$ $(i=1,2,\dots,m)$. Therefore $f_i(p)\subseteq q_i$, $i=1,2,\dots,m$. Since $\mathrm{diam}(q_i)<\epsilon$, for every $i=1,2,\dots,m$ and $x\in p$, $d(f_i(x),g_i(x))<\epsilon$. Since $p\in P$ was arbitrary, this shows $(f_1,\dots,f_m)\in U$. □

Proposition 2.3.14. *The conjugacy class of $(F_1^{\mathbb{L}},\dots,F_m^{\mathbb{L}})$ is dense in $H(\mathbb{L})^m=H(2^{\mathbb{N}})^m$.*

Proof. For a partition P and a tuple of surjective relations $(s_1^P,\dots,s_m^P)$ on P we consider

$$D(P,s_1^P,\dots,s_m^P)=\{(f_1,\dots,f_m)\in H(\mathbb{L})^m\colon \exists g\ (g^{-1}f_1g,\dots,g^{-1}f_mg)\in[P,s_1^P,\dots,s_m^P]\}.$$

Let D be the intersection of all sets of the form $D(P,s_1^P,\dots,s_m^P)$. From Lemma 2.3.13 it follows that if $(f_1,\dots,f_m)\in D$, then it has a dense conjugacy class.

We show that $(F_1^{\mathbb{L}},\dots,F_m^{\mathbb{L}})\in D$. Fix a partition P and a tuple $(s_1^P,\dots,s_m^P)$ of surjective relations on P. From the projective universality of the limit and the coinitiality of $\mathcal{F}$ in $\mathcal{F}_0$, there are a partition Q and an isomorphism $i\colon(P,s_1^P,\dots,s_m^P)\to(Q,F_1^{\mathbb{L}}\restriction Q,\dots,F_m^{\mathbb{L}}\restriction Q)$. Now take any $g\in H(\mathbb{L})$ that extends i, and notice that $(g^{-1}F_1^{\mathbb{L}}g,\dots,g^{-1}F_m^{\mathbb{L}}g)\in[P,s_1^P,\dots,s_m^P]$. □

Proposition 2.3.15. *The conjugacy class of $(F_1^{\mathbb{L}},\dots,F_m^{\mathbb{L}})$ is a G_δ in $H(\mathbb{L})^m=H(2^{\mathbb{N}})^m$.*

Proof. We show that the set of $(f_1, \dots, f_m) \in H(2^{\mathbb{N}})^m$ such that $(2^{\mathbb{N}}, f_1, \dots, f_m)$ satisfies (L1), the extension property, and (L2), is a G_δ. From Proposition 2.1.2 (ii), these are exactly structures that are isomorphic to the projective Fraïssé limit $(\mathbb{L}, F_1^{\mathbb{L}}, \dots, F_m^{\mathbb{L}})$, that is, structures that are conjugate to $(\mathbb{L}, F_1^{\mathbb{L}}, \dots, F_m^{\mathbb{L}})$.

1. Given $A \in \mathcal{F}$, we notice that

$$U_A = \{(f_1, \dots, f_m) \in H(2^{\mathbb{N}})^m \colon \text{ there is an epimorphism from } (2^{\mathbb{N}}, f_1, \dots, f_m) \text{ onto } A\}$$

is open.

2. Given $A = (A_0, s_1^A, \dots, s_m^A), B = (B_0, s_1^B, \dots, s_m^B) \in \mathcal{F}$, $\phi \colon B \to A$, and a continuous surjection $g \colon 2^{\mathbb{N}} \to A_0$, consider

$$\begin{aligned} E_{\phi,g} = &\{(f_1, \dots, f_m) \in H(2^{\mathbb{N}})^m \colon \text{ if } g \colon (2^{\mathbb{N}}, f_1, \dots, f_m) \to A \text{ is an epimorphism,} \\ &\text{then there is } h \colon (2^{\mathbb{N}}, f_1, \dots, f_m) \to B \text{ such that } g = \phi \circ h\}. \end{aligned}$$

We show that this set is open.

For A and $g \colon 2^{\mathbb{N}} \to A_0$ as above we define

$$H(g, A) = \{(f_1, \dots, f_m) \in H(2^{\mathbb{N}})^m \colon g \colon (2^{\mathbb{N}}, f_1, \dots, f_m) \to A \text{ is an epimorphism}\}.$$

This is a clopen set in $H(2^{\mathbb{N}})^m$. Therefore

$$E_{\phi,g} = \Big(H(2^{\mathbb{N}})^m \setminus H(g, A)\Big) \cup \left(\bigcup_h H(h, B)\right),$$

where the union is taken over continuous surjections $h \colon 2^{\mathbb{N}} \to B_0$ such that $g = \phi \circ h$, is an open set. Since there are only countably many clopen decompositions of $2^{\mathbb{N}}$, there are only countably many continuous surjections $g \colon 2^{\mathbb{N}} \to A_0$.

3. Clearly, every $(2^{\mathbb{N}}, f_1, \dots, f_m)$ satisfies (L2).

Hence,

$$\left(\bigcap_A U_A\right) \cap \left(\bigcap_{\phi,g} E_{\phi,g}\right)$$

is a G_δ set. It consists exactly of $(f_1, \ldots, f_m) \in H(2^{\mathbb{N}})^m$ such that $(2^{\mathbb{N}}, f_1, \ldots, f_m)$ satisfies (L1), the extension property, and (L2).

□

Proof of Theorem 2.3.1. This follows from Theorems 2.3.2, 2.3.7, and 2.3.9. □

2.4 Measure preserving homeomorphisms of the Cantor set

In this section we give examples of measures on the Cantor set such that a generic measure preserving homeomorphism exists and is realized as a projective Fraïssé limit. The main result of this section is Theorem 2.4.6.

Define a family

$$\begin{aligned}\mathcal{A} =&\{(A, \mu^A, F^A) \colon A \text{ is a compact zero-dimensional second countable topological space,}\\ &\mu^A \text{ is a probability measure on } A, F^A \text{ is a } \mu^A\text{-preserving homeomorphism of } A\}.\end{aligned}$$

Let $(A, \mu^A, F^A), (B, \mu^B, F^B) \in \mathcal{A}$. We say that $\phi \colon (B, \mu^B, F^B) \to (A, \mu^A, F^A)$ is an epimorphism if it is a continuous measure preserving surjection satisfying $\phi \circ F^B(b) = F^A \circ \phi(b)$ for every $b \in B$.

Denote by $\mathcal{C}$ the category whose collection of objects is equal to $\mathcal{A}$ and epimorphisms are as described above. For this category, and any subcategory of it, we define, in an obvious way, a projective Fraïssé family and a projective Fraïssé limit.

For a multiplicative subgroup of real numbers G, let

$$\begin{aligned}\mathcal{F}_G =&\{(A, \mu^A, F^A) \in \mathcal{A}^G \colon A \text{ is finite,}\\ &\mu^A \text{ is a probability measure on } A \text{ such that for every } a \in A, \mu^A(a) \in G,\\ &F^A \text{ is a } \mu^A\text{-preserving bijection of } A\}.\end{aligned}$$

Proposition 2.4.1. *The family $\mathcal{F}_G$ is a projective Fraïssé family.*

Proof. JEP: Take $(A, \mu^A, F^A), (B, \mu^B, F^B) \in \mathcal{F}_G$. Then $(A \times B, \mu^A \times \mu^B, F^A \times F^B)$ together with projections work.

AP: Take $(A, \mu^A, F^A), (B, \mu^B, F^B), (C, \mu^C, F^C) \in \mathcal{F}_G$, $\phi_1 \colon (B, \mu^B, F^B) \to (A, \mu^A, F^A)$ and

$\phi_2\colon (C,\mu^C,F^C) \to (A,\mu^A,F^A)$. Then (D,μ^D,F^D), where

$$D = \{(b,c) \in B \times C \colon \phi_1(b) = \phi_2(c)\},$$

μ^D is such that $\mu^D(b,c) = \frac{\mu^B(b)\mu^C(c)}{\mu^A(\phi_1(b))}$, and $F^D = F^B \times F^C$, together with projections $\phi_3\colon D \to B$ and $\phi_4\colon D \to C$ work. $\square$

It can be checked that the proof of Theorem 2.1.1, given in [21], can be adapted to show the existence and the uniqueness of the projective Fraïssé limit of $\mathcal{F}_G$. This checking reduces to verifying that the two lemmas stated below are true in our generalized setting. The proof of each of this lemmas is straightforward and we omit it.

Lemma 2.4.2. *Let $A,B,C \in \mathcal{F}_G$. Let $f\colon B \to A$, $g\colon C \to A$ and $\phi\colon C \to B$ be functions such that $g = f \circ \phi$. Assume that ϕ is an epimorphism. Then f is an epimorphism iff g is an epimorphism.*

Lemma 2.4.3. *Inverse limits of structures in $\mathcal{F}_G$ exist in the category $\mathcal{C}$.*

Fix G, a multiplicative subgroup of real numbers. Let $\mathbb{L} = (\mathbb{L},\mu^{\mathbb{L}},F^{\mathbb{L}})$ denote the projective Fraïssé limit of $\mathcal{F}_G$ (it will be clear all the time which G we are working with, therefore we will be writing $\mathbb{L}$ rather that $\mathbb{L}_G$). We concentrate now on the existence of a comeager conjugacy class in $H(\mathbb{L},\mu^{\mathbb{L}})$, the group of all measure preserving homeomorphisms of $\mathbb{L}$ (in fact, the underlying set $\mathbb{L}$ will typically be the Cantor set). In Proposition 2.3.15 we show that the conjugacy class of $F^{\mathbb{L}}$ is always a G_δ.

For a partition P and a $\mu^{\mathbb{L}} \restriction P$ preserving bijection F^P on P define

$$[P,\mu^{\mathbb{L}} \restriction P, F^P] = \{f \in H(\mathbb{L},\mu^{\mathbb{L}})\colon \forall p,q \in P\ f(p) = q \iff F^P(p) = q\}.$$

Similarly as in Proposition 2.3.15, we have the following.

Proposition 2.4.4. *The conjugacy class of $F^{\mathbb{L}}$ is a G_δ in $H(\mathbb{L},\mu^{\mathbb{L}})$.*

Proof. We show that the set of measure preserving homeomorphisms that satisfy (L1), the extension property, and (L2) is a G_δ.

1. Let $A \in \mathcal{F}_G$. Notice that

$$U_A = \{f \in H(\mathbb{L}, \mu^{\mathbb{L}})\colon \text{ there is an epimorphism from } (\mathbb{L}, \mu^{\mathbb{L}}, f) \text{ onto } A\}$$

is open.

2. Given $A = (A_0, \mu^A, F^A), B = (B_0, \mu^B, F^B) \in \mathcal{F}_G$, $\phi\colon B \to A$, and a continuous measure preserving surjection $g\colon (\mathbb{L}, \mu^{\mathbb{L}}) \to (A_0, \mu^A)$, consider

$$\begin{aligned} E_{\phi,g} =& \{f \in H(\mathbb{L}, \mu^{\mathbb{L}})\colon \text{ if } g\colon (\mathbb{L}, \mu^{\mathbb{L}}, f) \to A \text{ is an epimorphism,} \\ & \text{then there is } h\colon (\mathbb{L}, \mu^{\mathbb{L}}, f) \to B \text{ such that } g = \phi \circ h\}. \end{aligned}$$

We show that this set is open.

For A and $g\colon (\mathbb{L}, \mu^{\mathbb{L}}) \to (A_0, \mu^A)$ as above we define

$$H(g, A) = \{f \in H(\mathbb{L}, \mu^{\mathbb{L}})\colon g\colon (\mathbb{L}, \mu^{\mathbb{L}}, f) \to A \text{ is an epimorphism}\}.$$

This is a clopen set in $H(\mathbb{L}, \mu^{\mathbb{L}})$. Therefore

$$E_{\phi,g} = \Big(H(\mathbb{L}, \mu^{\mathbb{L}}) \setminus H(g, A)\Big) \cup \left(\bigcup_h H(h, B)\right),$$

where the union is taken over continuous measure preserving surjections $h\colon (\mathbb{L}, \mu^{\mathbb{L}}) \to (B_0, \mu^B)$ such that $g = \phi \circ h$, is an open set. Since there are only countably many clopen decompositions of $\mathbb{L}$, there are only countably many continuous measure preserving surjections $h\colon (\mathbb{L}, \mu^{\mathbb{L}}) \to (B_0, \mu^B)$.

3. For a clopen partition P let

$$R_P = \bigcup_{Q, F^Q} [Q, \mu^{\mathbb{L}} \restriction Q, F^Q],$$

where the union is taken over all clopen subpartitions Q of P and all $\mu^{\mathbb{L}} \restriction Q$ preserving bijection F^Q on Q.

Hence

$$\mathbb{F} = \left(\bigcap_A U_A\right) \cap \left(\bigcap_{\phi,g} E_{\phi,g}\right) \cap \left(\bigcap_P R_P\right)$$

is a G_δ set.

The set $\mathbb{F}$ consists exactly of $f \in H(\mathbb{L}, \mu^{\mathbb{L}})$ such that $(\mathbb{L}, \mu^{\mathbb{L}}, f)$ satisfies (L1), the extension property, and (L2), and these are exactly structures that are isomorphic to the projective Fraïssé limit $(\mathbb{L}, \mu^{\mathbb{L}}, F^{\mathbb{L}})$, that is, structures that are conjugate to $(\mathbb{L}, \mu^{\mathbb{L}}, F^{\mathbb{L}})$.

□

Below, we give examples of G such that the projective Fraïssé limit $F^{\mathbb{L}}$ is a generic measure preserving homeomorphism. First, we need a proposition.

Proposition 2.4.5. *Suppose that the collection of clopen sets $[P, \mu^{\mathbb{L}} \restriction P, F^P]$ form a π-basis (that is, for every open set U there is some $[P, \mu^{\mathbb{L}} \restriction P, F^P]$ such that $[P, \mu^{\mathbb{L}} \restriction P, F^P] \subseteq U$). Then the conjugacy class of $F^{\mathbb{L}}$ is dense in $H(\mathbb{L}, \mu^{\mathbb{L}})$.*

Proof. Fix an open set $U \subseteq H(\mathbb{L}, \mu^{\mathbb{L}})$. Let $[P, \mu^{\mathbb{L}} \restriction P, F^P] \subseteq U$. From the projective universality there is a partition Q and an isomorphism $i\colon (P, \mu^{\mathbb{L}} \restriction P, F^P) \to (Q, \mu^{\mathbb{L}} \restriction Q, F^{\mathbb{L}} \restriction Q)$. Now take any $g \in H(\mathbb{L}, \mu^{\mathbb{L}})$ that is an extension of i, and notice that $g^{-1}F^{\mathbb{L}}g \in [P, \mu^{\mathbb{L}} \restriction P, F^P] \subseteq U$. □

Summarizing, we have shown the following theorem.

Theorem 2.4.6. *Suppose that G is a multiplicative subgroup of real numbers and moreover suppose that the collection of clopen sets $[P, \mu^{\mathbb{L}} \restriction P, F^P]$ form a π-basis. Then there is $F \in H(\mathbb{L}, \mu^{\mathbb{L}})$ with a comeager conjugacy class. Moreover, F can be realized as a projective Fraïssé limit of $\mathcal{F}_G$.*

Example 2.4.7. Let M be a subset of positive natural numbers. Let

$$G = \{\frac{1}{m_1^{l_1} m_2^{l_2} \dots m_n^{l_n}} : m_1, m_2, \dots, m_n \in M, l_1, l_2, \dots, l_n \in \mathbb{Z}\}.$$

Then $\mathcal{F}_G$ satisfies the hypotheses of Theorem 2.4.6.

Proof. We show that the collection of clopen sets $[P, \mu^{\mathbb{L}}, F^P]$ is a π-basis. Fix U, take an arbitrary $h \in U$, and take an $\epsilon > 0$ such that $B(h, \epsilon) \subseteq U$ (here $B(h, \epsilon)$ denotes the ball centered at h

with radius ϵ). Fix a clopen partition Q such the diameter of each clopen is $< \epsilon$. Next, consider the finer partition $Q' = \{h^{-1}(A) \cap B \cap h(C) \colon A, B, C \in Q\}$. Note that the clopen of the form $h^{-1}(A) \cap B$ is mapped by h onto the clopen $A \cap h(B)$. Let P be a refinement of Q' into clopens of equal measure. We note that the number of clopens from P that are in $h^{-1}(A) \cap B$ is equal to the number of clopens from P that are in $A \cap h(B)$. Let F^P be a bijection of P such that for every $K_1, K_2 \in P$ and $A, B \in Q$, if $F^P(K_1) = K_2$ and $K_1 \subseteq h^{-1}(A) \cap B$, then $K_2 \subseteq A \cap h(B)$. We conclude $[P, \mu^{\mathbb{L}} \restriction P, F^P] \subseteq B(h, \epsilon) \subseteq U$.

$\square$

Let us compare our results to known results about the existence of a generic measure preserving homeomorphism of the Cantor set. Call a measure $\mu^{\mathbb{L}}$ *rational* when it is obtained when $M = \mathbb{N} \setminus \{0\}$, call it *dyadic rational* when $M = \{2\}$ (the notation comes from Example 2.4.7). Kechris and Rosendal [24] showed that there exists a comeager conjugacy class when μ is a dyadic rational or rational measure. Akin [1] showed that for every good and $\mathbb{Q}$-like measure μ, there is a comeager conjugacy class. We say that a Borel probability measure μ on $2^{\mathbb{N}}$ is *good* if for every clopen sets U and V in $2^{\mathbb{N}}$ such that $\mu(U) < \mu(V)$, there is a clopen subset U_1 of V such that $\mu(U) = \mu(U_1)$. We say that it is $\mathbb{Q}$-*like* if $\{\mu(U) \colon U \text{ is a clopen }\} + \mathbb{Z}$ is a $\mathbb{Q}$ vector subspace of $\mathbb{R}$. In particular, the rational measure is good and $\mathbb{Q}$-like.

Chapter 3

Isometry groups of locally compact separable metric spaces

3.1 Boolean actions

Let $(X, \mathcal{B}(X), \mu)$ be a standard Lebesgue space (i.e., there is a Polish topology on X whose family of Borel sets is $\mathcal{B}(X)$ and μ is a Borel probability measure on $\mathcal{B}(X)$). For $B \in \mathcal{B}(X)$, let $[B]_\mu$ be the μ-equivalence class of B. By $\mathcal{B}(X)/\mu$ we denote the Boolean algebra of all $[B]_\mu$, $B \in \mathcal{B}(X)$, with the usual Boolean operations. Let $\mathrm{Aut}(\mu)$ denote the Polish group of all measure preserving automorphisms of $(X, \mathcal{B}(X), \mu)$. We view $\mathrm{Aut}(\mu)$ as a closed subgroup of the orthogonal group $\mathrm{O}(L^2(\mu))$, where the latter group is equipped with the strong operator topology, by associating with $T \in \mathrm{Aut}(\mu)$ an orthogonal operator $O_T \in \mathrm{O}(L^2(\mu))$ given, with some abuse of notation, by

$$O_T(f) = f \circ T^{-1}.$$

(By $L^2(\mu)$ we understand the real valued $L^2(\mu)$.)

Let G be a Polish group. Assume we are given a continuous homomorphism $G \to \mathrm{Aut}(\mu)$, which we will view as a continuous action of G on $\mathcal{B}(X)/\mu$:

$$G \times \mathcal{B}(X)/\mu \ni (g, [B]_\mu) \to g \cdot [B]_\mu \in \mathcal{B}(X)/\mu.$$

We call such an action a (measure preserving) *Boolean action of G on $\mathcal{B}(X)/\mu$*. By a *spatial model* of such a Boolean action we mean a Borel action $G \times X \to X$ of G on X such that for each $B \in \mathcal{B}(X)$ and $g \in G$, we have

$$[g(B)]_\mu = g \cdot [B]_\mu.$$

Let us introduce one more notion. Let G be a Polish group and let $(X, \mathcal{B}(X), \mu)$ be a standard

Lebesgue space. By a *near-action* of G on $(X, \mathcal{B}(X), \mu)$ we mean a Borel map $G \times X \to X$; $(g, x) \to g \cdot x$ with the following properties:

(1) for the identity element of $1 \in G$, $1 \cdot x = x$ for almost every x;

(2) for each pair $g, h \in G$, $g \cdot (h \cdot x) = (gh) \cdot x$ for almost every x;

(3) each $g \in G$ preserves the measure μ.

Glasner and Weiss [15] showed the following proposition.

Proposition 3.1.1 (Glasner-Weiss)**.** *The following three notions are equivalent:*

(1) a near action of G on $(X, \mathcal{B}(X), \mu)$;

(2) a continuous homomorphism from G into $\mathrm{Aut}(\mu)$*;*

(3) a Boolean action of G on $(X, \mathcal{B}(X), \mu)$.

3.2 Lie groups

Recall that a *Lie group* is a topological group which is also a manifold, and such that the group operations are smooth.

In Proposition 3.2.1 we collect some well known properties of Lie groups that will be used. Important to us will be the notion of dimension of a Lie group, which can be understood as the linear dimension of its Lie algebra or, equivalently, as the dimension of the underlying manifold.

Proposition 3.2.1. *(i) Connected components of a Lie group are open and the connected component of the identity is a Lie group.*

(ii) If M is a Lie group and N a closed subgroup of M, then N is a Lie group; if, additionally, N is normal, then M/N is a Lie group.

(iii) Let M, N be Lie groups and let $f\colon M \to N$ be a continuous homomorphism. If f is injective, then $\dim(M) \leq \dim(N)$*; if f is surjective, then* $\dim(M) \geq \dim(N)$.

(iv) Let M, N be Lie groups with $\dim(M) = \dim(N)$ *and with N connected. If $f : M \to N$ is a continuous injective homomorphism, then f is surjective.*

Proof. (i) The first statement is clear from the definition of manifold. The second statement follows from the first one and the general fact that the connected component of the identity of a topological group is a subgroup.

(ii) See [41, Theorem 3.42] for the proof that N is Lie and [41, Theorem 3.64] for the proof that M/N is Lie.

(iii) This point follows from [41, Theorem 3.32].

(iv) By [41, Theorem 3.32], $f(M)$ is an open, so closed and open, subgroup of N. Since N is connected, $f(M) = N$. □

We say that a locally compact Polish group G' is *Lie projective* if for every open $U \subseteq G'$ there is a compact normal subgroup K of G' such that $K \subseteq U$ and G'/K is a Lie group. We will use the following deep theorem about locally compact groups.

Theorem 3.2.2. *[33, Theorem 4.6, p.175, Lemma 2.3.1, p.54] Suppose that G is a locally compact group. Then there is an open subgroup $G' < G$ that is Lie projective.*

A second countable group G is *pro-Lie* if it is Polish and each neighborhood of the identity contains a normal subgroup N such that G/N is Lie. In [27] we gave a short and self-contained proof of the following result of Hofmann and Morris [19] in the case of second countable groups.

Theorem 3.2.3 (Hofmann-Morris)**.** *A closed subgroup of a countable product of Lie groups is pro-Lie.*

All ideas in our proof of Theorem 3.2.3 come from Lemma 3.3.3 below.

3.3 A characterization of isometry groups

The goal of this section is to present a proof of the following characterization of groups of isometries of locally compact separable metric spaces. We will use this characterization in the next section to prove Theorem 3.4.1. The theorem below is a restatement of Theorem 1.3.2 stated in the introduction. Recall from the introduction that a topological group G is called a *group of isometries of X* if there exists an isomorphism that is also a homeomorphism between G and a subgroup of $\mathrm{Iso}(X)$.

Theorem 3.3.1. *Let G be a Polish group. Then G is a group of isometries of a locally compact separable metric space if and only if each neighborhood of the identity contains a closed subgroup H such that the space G/H is locally compact and $N(H)$ is open.*

We state explicitly the property from Theorem 3.3.1:

($*$) for every open neighborhood of the identity $U \subseteq G$ there is a closed subgroup $H < G$ such that $H \subseteq U$, $N(H)$ is open, and G/H is locally compact.

We will be later referring to this property of G as *property* ($*$).

First we show that if a Polish group G_0 has property ($*$), then it has the following stronger version of that property.

Lemma 3.3.2. *Suppose a Polish group G_0 has property ($*$), then each neighborhood of* 1 *contains a closed group H such that $N(H)$ is open and $N(H)/H$ is a Lie group.*

Proof. To prove this property let $U \ni 1$ be open. Let $V \ni 1$ be open such that

$$V^2 \subseteq U. \tag{3.3.1}$$

By property ($*$) there exists a closed group

$$H_0 \subseteq V \tag{3.3.2}$$

such that $N(H_0)$ is open and $N(H_0)/H_0$ is locally compact. By Theorem 3.2.2 there exists an open subgroup N of $N(H_0)$ such that $H_0 < N$ and N/H_0 is Lie projective. We can assume without loss of generality (by decreasing V) that $V \subseteq N$. Let $\pi \colon N(H_0) \to N(H_0)/H_0$ be the quotient homomorphism. Then $\pi(V)$ is open. Take

$$K \subseteq \pi(V) \tag{3.3.3}$$

a compact normal subgroup of N/H_0 such that $(N/H_0)/K$ is Lie. Let

$$H = \pi^{-1}(K).$$

By (3.3.1), (3.3.2), (3.3.3), $H \subseteq U$. Since K is normal in N/H_0, H is normal in N, and so we have $N(H) \supseteq N$, implying that $N(H)$ is open. Finally, since we have the homeomorphism

$$N(H)/H = N(H)/\pi^{-1}(K) \cong (N(H)/H_0)/K,$$

there is an open subgroup of $N(H)/H$ that is isomorphic to the Lie group $(N/H_0)/K$. Thus, $N(H)/H$ is Lie. □

Lemma 3.3.3. *A closed subgroup of a Polish group with property* $(*)$ *has property* $(*)$.

Proof. It may be beneficial first to follow this proof in the concrete setting where one assumes that the subgroup in question is a closed subgroup of a countable product of Lie groups and one argues that it has property $(*)$. Many of the essential difficulties of the proof remain in this special case.

Let a Polish group G_0 with property $(*)$ be given. Then G_0 has the stronger version of property $(*)$ stated above. Fix closed subgroups $H_n < G_0$ and open subgroups $M_n < G_0$, $n \in \mathbb{N}$, so that $H_n \subseteq M_n$ and H_n is normal in M_n, M_n/H_n is a Lie group, and each open neighborhood of 1 in G_0 contains H_n for all but finitely many n. We assume, as we can, that $M_{n+1} \subseteq M_n$. Let π_n be the product of the natural quotient functions $G_0 \to G_0/H_i$

$$\pi_n : G_0 \to \prod_{i \leq n} G_0/H_i,$$

and let $\pi_{n,N}$, $n \leq N$, be the projection

$$\pi_{n,N} : \prod_{i \leq N} G_0/H_i \to \prod_{i \leq n} G_0/H_i.$$

Note that for $N \geq n$

$$\pi_{n,N} \circ \pi_N = \pi_n.$$

Let $L_n = \prod_{i \leq n} M_i/H_i$. We point out that L_n is a Lie group and that

$$\pi_n \restriction M_n : M_n \to L_n \text{ and } \pi_{n,N} \restriction L_N : L_N \to L_n.$$

are continuous group homomorphisms.

Let $G < G_0$ be closed. We will show that G has property $(*)$. Fix an open neighborhood of 1 in G. Find n such that $G \cap H_n$ is included in that neighborhood. Note that the normalizer in G of $G \cap \bigcap_{i \leq n} H_i$ includes $G \cap M_n$ and is, therefore, open in G. Thus, to prove that G has property $(*)$, it will suffice to show that for each n, $G/(G \cap \bigcap_{i \leq n} H_i)$ is locally compact.

By Proposition 3.2.1(ii), the closure in L_N of the subgroup $\pi_N(G \cap M_N)$ is a Lie group, and, by Proposition 3.2.1(i), the connected component of the identity of this closure is a Lie group as well. So if we let

$$A_N = \text{ the connected component of 1 of } \overline{\pi_N(G \cap M_N)},$$

then A_N is a Lie group. For $n \leq N$, let

$$B_{n,N} = \ker\left(\pi_{n,N} \restriction A_N\right).$$

Note that since $M_{N+1} \subseteq M_N$, $\pi_{N,N+1}(A_{N+1})$ is a connected subgroup of $\overline{\pi_N(G \cap M_N)}$, and therefore we have

$$\pi_{N,N+1}(A_{N+1}) \subseteq A_N. \tag{3.3.4}$$

For $N \geq n$, $A_N/B_{n,N}$ is a Lie group by Proposition 3.2.1(ii). For these groups we have the following claim.

Claim 3.3.4. *For every n there is $i_n \geq n$ such that for $N \geq i_n$*

$$\dim\left(A_{i_n}/B_{n,i_n}\right) = \dim\left(A_N/B_{n,N}\right).$$

Proof. Let $N \geq n$. Inclusion (3.3.4) induces an injective continuous homomorphism

$$A_{N+1}/(\pi^{-1}_{N,N+1}(B_{n,N}) \cap A_{N+1}) \to A_N/B_{n,N}.$$

It follows by Proposition 3.2.1(iii) that

$$\dim(A_{N+1}/(\pi^{-1}_{N,N+1}(B_{n,N}) \cap A_{N+1})) \leq \dim(A_N/B_{n,N}). \tag{3.3.5}$$

Note however that

$$\pi_{N,N+1}^{-1}(B_{n,N}) \cap A_{N+1} \subseteq B_{n,N+1},$$

and, therefore, there is a surjective continuous homomorphism

$$A_{N+1}/(\pi_{N,N+1}^{-1}(B_{n,N}) \cap A_{N+1}) \to A_{N+1}/B_{n,N+1},$$

so by Proposition 3.2.1(iii) we have

$$\dim(A_{N+1}/B_{n,N+1}) \leq \dim(A_{N+1}/(\pi_{N,N+1}^{-1}(B_{n,N}) \cap A_{N+1})).$$

From this inequality and from (3.3.5) we get

$$\dim(A_{N+1}/B_{n,N+1}) \leq \dim(A_N/B_{n,N}).$$

We conclude that the natural number valued function

$$N \to \dim(A_N/B_{n,N}),$$

defined for $N \geq n$, is non-increasing, and the conclusion of the claim follows. □

For $n \in \mathbb{N}$, $i_n \geq n$ will denote the natural number from Claim 3.3.4.

Claim 3.3.5. *Let $n \in \mathbb{N}$. For $N \geq i_n$,*

$$\pi_{n,N+1}(A_{N+1}) = \pi_{n,N}(A_N).$$

Proof. The homomorphisms $\pi_{n,N} \upharpoonright A_N$ and $\pi_{n,N+1} \upharpoonright A_{N+1}$ induce injective continuous homomorphisms

$$\widehat{\pi}_{n,N} \colon A_N/B_{n,N} \to L_n \text{ and } \widehat{\pi}_{n,N+1} \colon A_{N+1}/B_{n,N+1} \to L_n.$$

Furthermore, from (3.3.4), we see that

$$\begin{aligned}\widehat{\pi}_{n,N+1}(A_{N+1}/B_{n,N+1}) &= \pi_{n,N+1}(A_{N+1}) \\ &\subseteq \pi_{n,N}(A_N) = \widehat{\pi}_{n,N}(A_N/B_{n,N}).\end{aligned} \tag{3.3.6}$$

Note that by Claim 3.3.4

$$\dim(A_{N+1}/B_{n,N+1}) = \dim(A_N/B_{n,N}),$$

and that $A_N/B_{n,N}$ is connected, as A_N is. Now since $\widehat{\pi}_{n,N}$ and $\widehat{\pi}_{n,N+1}$ are injective, by (3.3.6), we can consider the injective homomorphism

$$(\widehat{\pi}_{n,N})^{-1} \circ \widehat{\pi}_{n,N+1} \colon A_{N+1}/B_{n,N+1} \to A_N/B_{n,N}.$$

Since this homomorphism is Borel, it is continuous, and, by what was said above, Proposition 3.2.1(iv) implies that it is surjective. From this assertion and from (3.3.6), the conclusion of the claim follows immediately. □

By Claim 3.3.5, $\pi_{n,N}(A_N)$ does not depend on N as long as $N \geq i_n$. Put

$$C_n = \pi_{n,N}(A_N),$$

for any $N \geq i_n$.

Claim 3.3.6. *For every n,*

$$C_n \subseteq \pi_n(G \cap \bigcap_i M_i).$$

Proof. Note first that for each n, $\pi_{n,n+1}(C_{n+1}) = C_n$. This is because, for $N \geq i_n, i_{n+1}$,

$$\begin{aligned}C_n &= \pi_{n,N}(A_N) = \pi_{n,n+1}(\pi_{n+1,N}(A_N)) \\ &= \pi_{n,n+1}(C_{n+1}).\end{aligned} \tag{3.3.7}$$

We will use the following general observation concerning Polish groups, see [11, Corollary 2.2.2]. Let d_l be a left invariant metric on G_0, and let d_r be the right invariant metric on G_0 given by

$d_r(x,y) = d_l(x^{-1}, y^{-1})$. Then the metric d defined by

$$d = d_l + d_r$$

is a complete metric on G_0. We will also need the following definition. For each $i \in \mathbb{N}$ and $g_1, g_2 \in M_i$, let ρ_i be given by the formula

$$\begin{aligned}\rho_i(g_1 H_i, g_2 H_i) = &\inf\{d_l(g_1 h_1, g_2 h_2)\colon h_1, h_2 \in H_i\}\\ &+ \inf\{d_r(g_1 h_1, g_2 h_2)\colon h_1, h_2 \in H_i\}.\end{aligned}$$

Since H_i is a normal subgroup of M_i, by [11, Lemma 2.2.8], ρ_i is a metric on M_i/H_i inducing the quotient topology.

Fix n_0. Let y_0 be an arbitrary element of C_{n_0}. We will show that $y_0 \in \pi_{n_0}(G \cap \bigcap_i M_i)$. Using (3.3.7), we can recursively pick $c_n \in M_n/H_n$ so that for each n we have

$$(c_0, \ldots, c_n) \in C_n \text{ and } (c_0, \ldots, c_{n_0}) = y_0. \tag{3.3.8}$$

By definition of C_n and A_N we have

$$C_n \subseteq \pi_{n,N}\left(\overline{\pi_N(G \cap M_N)}\right) \subseteq \overline{\pi_n(G \cap M_n)},$$

where $N \geq i_n$ is arbitrary and where the closure is taken in L_n. Using this observation and (3.3.8), we can pick recursively on n a sequence $g_n \in G \cap M_n$, $n \in \mathbb{N}$, so that $\pi_n(g_n) = (g_n H_0, \ldots, g_n H_n)$ is as close to $(c_0, \ldots, c_n)$ as we wish, say, we wish that for all $i \leq n$

$$\rho_i(g_n H_i, c_i) < \frac{1}{n+1}. \tag{3.3.9}$$

Consider now an arbitrary left coset gH_i of H_i in M_i. Since $g \in M_i$, we have $gH_i = H_i g$. Thus, the d_l-diameter of gH_i and the d_r-diameter of gH_i are equal to the d_l- and the d_r-diameters of H_i, respectively. From this observation, from the definition of ρ_i, and from (3.3.9), it follows that for

$i \leq n$

$$\begin{aligned} d_l(g_i, g_n) &\leq d_l\text{-diam}(g_i H_i) + \frac{1}{i+1} + d_l\text{-diam}(c_i) + \frac{1}{n+1} + d_l\text{-diam}(g_n H_i) \\ &\leq \frac{2}{i+1} + 3 \cdot d\text{-diam}(H_i), \end{aligned}$$

and similarly

$$d_r(g_i, g_n) \leq \frac{2}{i+1} + 3 \cdot d\text{-diam}(H_i).$$

Thus, for $i \leq n$ we get that

$$d(g_i, g_n) \leq \frac{4}{i+1} + 6 \cdot d\text{-diam}(H_i),$$

and therefore the sequence (g_i) is d-Cauchy. Since d is complete, (g_i) converges to some g_∞. Since $M_{i+1} \subseteq M_i$ and since each M_i and G are closed, we see that

$$g_\infty \in G \cap \bigcap_i M_i.$$

Furthermore, from (3.3.9) we see that for each n

$$\pi_n(g_\infty) = (c_0, \ldots, c_n),$$

in particular, by (3.3.8), we have $\pi_{n_0}(g_\infty) = y_0$, as required. □

Now we are ready to finish the proof, that is, show that for every n,

$$G \Big/ \left(G \cap \bigcap_{i \leq n} H_i \right)$$

is locally compact. We will apply the following fact that is easy to prove using the Baire category theorem: if a homogeneous Polish space contains an open non-empty subset that is σ-compact, then the space is itself locally compact. (By a homogeneous space we mean a space in which for each pair of points x, y there is a homeomorphism of the space onto itself mapping x to y.) Now fix n and note that $G/\left(G \cap \bigcap_{i \leq n} H_i\right)$ is a Polish space with its natural quotient topology. It is homogeneous and it contains the Polish group

$$(G \cap M_n) \Big/ \left(G \cap \bigcap_{i \leq n} H_i \right)$$

as an open subset. Thus, it suffices to show that this last group is σ-compact.

Take $N \geq i_n$. Consider the following commutative diagram. The functions in the diagram are defined below it.

$$\begin{tikzcd} A_N \arrow[r, "\pi_{n,N} \restriction A_N"] \arrow[dr, "\tau"'] & \pi_n (G \cap M_n) \\ & (G \cap M_n) / \left(G \cap \bigcap_{i \leq n} H_i \right) \arrow[u, "\rho"'] \end{tikzcd}$$

The range of the continuous homomorphism $\pi_{n,N} \restriction A_N$ is included in the group $\pi_n (G \cap M_n)$ by Claim 3.3.6. The function ρ is defined as follows. In general, given groups M, $N \triangleleft M$, $N_0, \ldots, N_n \triangleleft M$ and $K < M$, there exist natural injective homomorphisms

$$K/(K \cap N) \to M/N \text{ and } M/\bigcap_{i \leq n} N_i \to \prod_{i \leq n} M/N_i.$$

From these general principles, we get two continuous injective homomorphisms

$$(G \cap M_n) \Big/ \left(G \cap \bigcap_{i \leq n} H_i \right) \to M_n / \bigcap_{i \leq n} H_i$$

and

$$M_n / \bigcap_{i \leq n} H_i \to \prod_{i \leq n} M_i / H_i = L_n,$$

and we let ρ be their compositions. Clearly ρ is a continuous injective homomorphism. By tracing the definitions it is easy to see that ρ is onto $\pi_n (G \cap M_n)$; thus, ρ is a continuous isomorphism. We define τ by

$$\tau = (\rho)^{-1} \circ \pi_{n,N}.$$

The function τ is a homomorphism and it is Borel, since ρ^{-1} is Borel being the inverse of a continuous injection. Since τ maps a Polish group A_N into a second countable group, it is a continuous homomorphism.

Since A_N is locally compact, it is σ-compact, and, therefore, so is $\tau(A_N)$ by continuity of τ. Thus, it suffices to show that $\tau(A_N)$ has countable index in

$$(G \cap M_n) \Big/ \left(G \cap \bigcap_{i \le n} H_i \right)$$

Since ρ is an isomorphism, it follows from the diagram that it is enough to prove that $\pi_{n,N}(A_N)$ has countable index in $\pi_n (G \cap M_n)$.

To prove this assertion note that from the definition of A_N and from Proposition 3.2.1(i), A_N is a non-empty relatively open subset of

$$\overline{\pi_N (G \cap M_N)},$$

which allows us to pick $g_j \in G \cap M_N$, $j \in \mathbb{N}$, so that

$$\bigcup_j \pi_N(g_j) A_N = \overline{\pi_N (G \cap M_N)}.$$

Applying $\pi_{n,N}$ to both sides of the equality above, removing the closure operation, and using $\pi_{n,N} \circ \pi_N = \pi_n$ and the fact that $\pi_{n,N}$ restricted to the group L_N is a homomorphism, we obtain

$$\bigcup_j \pi_n(g_j) \pi_{n,N}(A_N) \supseteq \pi_n (G \cap M_N)$$

with $\pi_n(g_j) \in \pi_n(G \cap M_n)$. Thus, $\pi_{n,N}(A_N)$ will have countable index in $\pi_n(G \cap M_n)$ as soon as we show that the index of $\pi_n (G \cap M_N)$ in $\pi_n (G \cap M_n)$ is countable. But this last assertion is obvious since $\pi_n \restriction (G \cap M_n)$ is a homomorphism and $G \cap M_N$ is open and, therefore, of countable index in $G \cap M_n$. $\square$

Lemma 3.3.7. *A countable product of Polish groups with property* $(*)$ *has property* $(*)$.

Proof. Assume G_n, $n \in \mathbb{N}$, are Polish groups with property $(*)$. Fix an open neighborhood of 1 in

$\prod_n G_n$, which we can assume to be of the form

$$U_0 \times \cdots \times U_m \times \prod_{n>m} G_n.$$

Let $H_i < G_i$, $i \leq m$, be closed and such that $H_i \subseteq U_i$, G_i/H_i locally compact, and $N(H_i)$ open. Then the subgroup of $\prod_n G_n$

$$H = H_0 \times \cdots \times H_m \times \prod_{n>m} G_n.$$

is as required, that is, H is included in the given neighborhood of 1, the space $(\prod_n G_n)/H$ is locally compact, and $N(H)$ is open. $\square$

Proof of $\Leftarrow$ in Theorem 3.3.1. By the theorem of Gao and Kechris [12, Theorem 6.3], and by Lemmas 3.3.3 and 3.3.7, it suffices to show that groups G of the form

$$G = S_\infty \ltimes M^{\mathbb{N}}$$

have property $(*)$, where M is Polish locally compact and S_∞ acts by automorphisms on $M^{\mathbb{N}}$ as follows

$$\sigma(h)(i) = h(\sigma^{-1}(i)),$$

where $\sigma \in S_\infty$ and $h \in M^{\mathbb{N}}$. To prove this fact, fix an open neighborhood of $1 \in G$, which we can assume to be of the form

$$\{\sigma \in S_\infty : \sigma \restriction n = \mathrm{id}\} \times U^n \times M^{\mathbb{N}\setminus n}$$

for some open $1 \in U \subseteq M$ and some $n \in \mathbb{N}$, where n denotes the set $\{0, \ldots, n-1\}$. Let

$$H = \{\sigma \in S_\infty : \sigma \restriction n = \mathrm{id}\} \ltimes \left(\{1\}^n \times M^{\mathbb{N}\setminus n}\right).$$

Clearly H is a closed subgroup of G contained in the given neighborhood of 1. Its normalizer $N(H)$ contains

$$\{\sigma \in S_\infty : \sigma \restriction n = \mathrm{id}\} \times M^{\mathbb{N}}$$

and therefore is open. The quotient G/H contains an open subset homeomorphic to M^n, which makes the space G/H locally compact. $\square$

Proof of $\Rightarrow$ of Theorem 3.3.1. Fix d, a left invariant metric on G. Without loss of generality d is bounded by 1. Fix a decreasing sequence U_n, $n \in \mathbb{N}$, of open neighborhoods of 1 in G such that $\mathrm{diam}(U_n) \to 0$ and $\bigcap_n U_n = \{1\}$. Let $H_n \subseteq U_n$ be closed subgroups of G as in property $(*)$. Set $M_n = N(H_n)$.

Consider $\oplus_n(G/H_n)$, the topological direct sum of the spaces G/H_n. This is a locally compact separable metrizable space. We define d^* by letting, for $x, y \in G$,

$$d^*(xH_n, yH_n) = \inf\{d(xu, yv) \colon u, v \in H_n\}$$

and, if $m \neq n$,

$$d^*(xH_n, yH_m) = 1.$$

We show that d^* is a metric on the set $\oplus_n(G/H_n)$. Note first that, by [11, Lemma 2.2.8], for each n, d^* restricted to M_n/H_n is a metric. This remains true, by left-invariance of d, for the restriction of d^* to $g(M_n/H_n)$, for each $g \in G$. Now, all the properties of a metric are clearly true of d^* except perhaps for the fact that $xH_n = yH_n$ whenever $d^*(xH_n, yH_n) = 0$. To show this, fix sequences $(u_i)_i$ and $(v_i)_i$ in H_n such that $d(xu_i, yv_i) \to 0$. Then $d(v_i^{-1}y^{-1}xu_i, 1) \to 0$, which allows us to fix i_0 such that $v_{i_0}^{-1}y^{-1}xu_{i_0} \in M_n$, implying $y^{-1}x \in M_n$. Hence x and y are in the same coset of M_n in G. Therefore $xH_n = yH_n$ by the remark at the beginning of this argument.

Notice that, by left invariance of d, for each $g \in G$ the function

$$\oplus_n(G/H_n) \ni c \to gc \in \oplus_n(G/H_n) \tag{3.3.10}$$

is an isometry with respect to d^*.

We show that d^* is compatible with the quotient topology on $\oplus_n(G/H_n)$. It suffices to see that both topologies agree on G/H_n, for each n. We start by showing that M_n/H_n is open in each of the two topologies. Clearly this set is open in the quotient topology. For the topology induced by d^* suppose that

$$d^*(x_iH_n, xH_n) \to 0,$$

where $x_i \notin M_n$ and $x \in G$. Suppose that $x \in M_n$. Since H_n is normal in M_n, for some sequences $(u_i)_i$ and $(v_i)_i$ in H_n, we have

$$d(x_i u_i, v_i x) \to 0,$$

i.e., $d(v_i^{-1} x_i u_i, x) \to 0$. Note that for each i, $v_i^{-1} x_i u_i \notin M_n$. Hence, since M_n is open in G, $x \notin M_n$, and we get a contradiction.

Since (3.3.10) is an isometry with respect to d^*, $g(M_n/H_n)$, where $g \in G$, is open in G/H_n and homeomorphic to M_n/H_n with respect to each of the two topologies. Thus, it suffices to see that the two topologies coincide on M_n/H_n, which follows immediately from [11, Lemma 2.2.8].

We show now that $f \colon G \to \mathrm{Iso}\,(\oplus_n (G/H_n))$ given by

$$f(g)(xH_n) = (gx)H_n,$$

for each n, is an embedding of Polish groups. Clearly f is a group homomorphism. Next note that f is injective. Indeed, let $g_1 \neq g_2$. Take n such that $g_2^{-1} g_1 \notin H_n$. Then $g_1 H_n \neq g_2 H_n$, i.e., $f(g_1)(H_n) \neq f(g_2)(H_n)$.

Finally, we show that f is a topological embedding. First let $g_i \to g$. We want to show that for each $x \in G$ and $n \in \mathbb{N}$, $d^*(g_i x H_n, g x H_n) \to 0$. By the definition of d^*, we have the inequality $d^*(g_i x H_n, g x H_n) \leq d(g_i x, gx)$, which yields the desired conclusion since $d(g_i x, gx) \to 0$.

Now suppose that $f(g_i) \to f(g)$. We want $g_i \to g$. Fix $\varepsilon > 0$. Take n such that $\mathrm{diam}(H_n) < \frac{\varepsilon}{3}$. By assumption we have $d^*(g_i H_n, g H_n) \to 0$, which allows us to choose sequences $(u_i)_i$ and $(v_i)_i$ in H_n such that $d(g_i u_i, g v_i) \to 0$. Take i_0 such that for $i \geq i_0$, $d(g_i u_i, g v_i) < \frac{\varepsilon}{3}$. Since

$$\begin{aligned} d(g_i, g) &\leq d(g_i, g_i u_i) + d(g_i u_i, g v_i) + d(g v_i, g) \\ &= d(1, u_i) + d(g_i u_i, g v_i) + d(v_i, 1), \end{aligned}$$

we get $d(g_i, g) < \varepsilon$, for $i \geq i_0$, and we are done. □

We conclude this section with the following corollary. This corollary is a restatement of Corollary 1.3.3 stated in the introduction.

Corollary 3.3.8. *Let G be a Polish group of isometries of a locally compact separable metric space,* *and let N be a closed normal subgroup of G. Then G/N is also a Polish group of isometries of a locally compact separable metric space. In other words, the class of Polish groups of isometries of locally compact separable metric spaces is closed under taking images of continuous homomorphisms onto Polish groups.*

By Theorem 3.3.1, it suffices to show that the property from this theorem holds for G/N if it holds for G. Let $\pi : G \to G/N$ be the quotient homomorphism. Let U be an open neighborhood of 1 in G/N. Let V be an open neighborhood of 1 in G/N whose closure is contained in U. Pick $H < \pi^{-1}(V)$ a closed subgroup of G with $N(H)$ open and with G/H locally compact. Define

$$H' = \overline{\pi(H)},$$

where the closure is taken in G/N. Obviously H' is a closed subgroup of G/N contained in U. The normalizer of H' in G/N contains $\pi(N(H))$ and therefore is open. The natural function

$$G/H \to G/\pi^{-1}(H') \cong (G/N)/H'$$

is surjective, continuous, and open. Thus $(G/N)/H'$ is locally compact.

3.4 Boolean actions of isometry groups

The goal of this section is to prove Theorem 3.4.1. This is a restatement of Theorem 1.3.1 stated in the introduction.

Theorem 3.4.1. *Let G be a Polish group of isometries of a locally compact separable metric space. Then each measure preserving Boolean action of G has a spatial model.*

Theorem 3.4.1 follows from Theorem 3.3.1 and from Lemma 3.4.3 below. Therefore, proving this lemma is all that remains to be done.

The following notion comes from [14]. It will be used in the proof of Lemma 3.4.3 through an application of Theorem 3.4.2 below. Assume we have a measure preserving Boolean action of a Polish group G on $\mathcal{B}(X)/\mu$. We say that a function $f \in L^\infty(\mu)$ is *G-continuous* if $f \circ g_n$ converges to f in the L^∞-norm whenever g_n converges to the identity in G. (By $L^\infty(\mu)$ we understand the real valued $L^\infty(\mu)$.)

Theorem 3.4.2. ([14, Theorem 2.2], see also [15, Theorem 1.7]) *A measure preserving Boolean action of G on $\mathcal{B}(X)/\mu$ admits a spatial model if and only if G-continuous functions are dense in $L^2(\mu)$.*

We note that since $\mathrm{Aut}(\mu)$ is a subgroup of $\mathrm{O}(L^2(\mu))$, each continuous homomorphism $G \to \mathrm{Aut}(\mu)$, i.e., Boolean action of G, gives rise to a continuous homomorphism $G \to \mathrm{O}(L^2(\mu))$, i.e., a continuous representation of G. We call this representation the *representation induced by the Boolean action.*

Lemma 3.4.3. *Let G be a group with property $(*)$. Then every Boolean action of G has a spatial model.*

Proof. Fix a standard Borel space $(X, \mathcal{B}(X), \mu)$ with a Borel probability measure μ and fix a Boolean action of G. We aim to show that G-continuous functions are dense in $L^2(\mu)$, which will prove the lemma by Theorem 3.4.2. We consider the representation of G induced by the Boolean action. The first step of the proof consisting of an application of the Ryll-Nardzewski fixed point theorem is borrowed from the proof of [15, Theorem 2.3]. Fix $f_0 \in L^2(\mu)$ and $\varepsilon > 0$. Consider

$$B(f_0, \varepsilon) = \{f \in L^2(\mu) \colon \|f - f_0\|_2 \leq \varepsilon\}.$$

Let $U_{f_0} \subseteq G$ be an open neighborhood of 1 such that $U_{f_0}.f_0 \subseteq B(f_0, \varepsilon)$. Since G has property $(*)$, we can fix a closed subgroup $H < G$ such that $H \subseteq U_{f_0}$, G/H is locally compact and $N(H)$ is open.

Since $L^2(\mu)$ is reflexive, by the Banach–Alaoglu theorem, the closed ball $B(f_0, \varepsilon)$ is weakly compact. Let

$$D = \overline{\mathrm{conv}\{h.f_0 \colon h \in H\}}^w$$

be the weak closure of the convex hull of the H-orbit of f_0 in $L^2(\mu)$.

The set D is a weakly compact convex H-invariant subset of $B(f_0, \varepsilon)$. By the Ryll-Nardzewski fixed point theorem, see [6, Chapter V, Theorem 10.8], there is $f_1 \in D$ which is H-fixed. Without loss of generality, we can assume that $f_1 \in L^\infty(\mu)$. (If f_1 is not bounded replace it by f^M satisfying

$f^M(x) = f_1(x)$ if $|f(x)| < M$ and $f^M(x) = M$ if $|f(x)| \geq M$, for some $M > 0$ such that $f^M \in B(f_0, \varepsilon)$.)

Let $\mathcal{A}$ be the smallest closed sublattice of $L^2(\mu)$ (i.e. $\mathcal{A}$ is a linear space closed under max and min) containing f_1, constant functions, and closed under the action of $N(H)$.

Our goal is to show that G-continuous functions are dense in $\mathcal{A}$ in the L^2-norm. Note that this will finish the proof since then, in particular, we get some G-continuous function f such that $\|f - f_1\|_2 < \varepsilon$. Since $\|f_0 - f_1\|_2 < \varepsilon$, where f_0 is the function we fixed at the beginning, we get $\|f - f_0\|_2 < 2\varepsilon$.

Let Q be a dense, countable subgroup of $N(H)$. For $h \cdot f_1 \in L^2(\mu)$ with $h \in Q$, let $f_{1,h} \colon X \to \mathbb{R}$ be a Borel function that is a representative of $h \cdot f_1$. Let $\mathcal{C}$ be the countable algebra of Borel subsets of X generated by the preimages under $f_{1,h}$, $h \in Q$, of rational intervals of $\mathbb{R}$. Let $\mathcal{B} \subseteq \mathcal{B}(X)$ be the σ-algebra generated by $\mathcal{C}$.

Claim 3.4.4. *For $h \in N(H)$ and $B \in \mathcal{B}$ we have*

$$h \cdot [B]_\mu \in \mathcal{B}/\mu.$$

Proof of Claim 3.4.4. The conclusion of the claim holds for all $h \in Q$ by the definition of $\mathcal{B}$. Now equip $\mathcal{B}(X)/\mu$ with the metric

$$\mathrm{dist}([B_1]_\mu, [B_2]_\mu) = \mu(B_1 \triangle B_2).$$

Note that since $\mathcal{B}$ is a σ-subalgebra of $\mathcal{B}(X)$, $\mathcal{B}/\mu$ is a closed subset of $\mathcal{B}(X)/\mu$. The action of $N(H)$ on $\mathcal{B}(X)/\mu$ is continuous. Since Q is dense in $N(H)$, the claim follows. $\square$

Define the function $\pi \colon X \to \{0,1\}^{\mathcal{C}}$ by

$$\pi(x)(B) = 1 \text{ iff } x \in B$$

for $B \in \mathcal{C}$. The function π is Borel. Set

$$X' = \pi(X),\ \mu' = \pi_*(\mu),$$

where, as usual, $\pi_*(\mu)(B) = \mu(\pi^{-1}(B))$ for $B \subseteq X'$ Borel. Note that there is a standard Borel set $Y \subseteq X'$ with $\mu'(Y) = 1$ in which case

$$\mathcal{B}(X')/\mu' \cong \mathcal{B}(Y)/(\mu' \restriction \mathcal{B}(Y)).$$

Thus, we can assume that X' itself is standard Borel. Note that X' comes equipped with the topology inherited from $\{0,1\}^{\mathcal{C}}$, which we will use.

The σ-algebra $\mathcal{B}$ consists precisely of preimages under π of elements of $\mathcal{B}(X')$. Mapping $B \in \mathcal{B}$ to the unique $B' \in \mathcal{B}(X')$ with $B = \pi^{-1}(B')$ induces a measure preserving isomorphism

$$\pi_* \colon \mathcal{B}/\mu \to \mathcal{B}(X')/\mu'. \tag{3.4.1}$$

By Claim 3.4.4, for each $B_0 \in \mathcal{B}(X')$ and $h \in N(H)$ there exists $B_1 \in \mathcal{B}(X')$ such that

$$h \cdot [\pi^{-1}(B_0)]_\mu = [\pi^{-1}(B_1)]_\mu,$$

which allows us to define

$$h \cdot [B_0]_{\mu'} = [B_1]_{\mu'}. \tag{3.4.2}$$

It is now easy to check that this is a measure preserving Boolean action of $N(H)$ on $\mathcal{B}(X')/\mu'$ and that for $h \in N(H)$

$$\pi_*(h \cdot [B]_\mu) = h \cdot \pi_*([B]_\mu). \tag{3.4.3}$$

The measure preserving isomorphism (3.4.1) induces in the natural way a lattice isomorphism

$$\pi_* \colon L^2(\mu \restriction \mathcal{B}) \to L^2(\mu').$$

This lattice isomorphism is also an isometry between $L^2(\mu \restriction \mathcal{B})$ and $L^2(\mu')$ and, when restricted to $L^\infty(\mu \restriction \mathcal{B})$, it is an isometry between $L^\infty(\mu \restriction \mathcal{B})$ and $L^\infty(\mu')$. Furthermore, note that for the representation of $N(H)$ in $\mathrm{O}(L^2(\mu \restriction \mathcal{B}))$ induced by its Boolean action on $\mathcal{B}/\mu$ given by Claim 3.4.4 and for the representation of $N(H)$ in $\mathrm{O}(L^2(\mathcal{B}(X')))$ induced by the Boolean action given by (3.4.2), we have by (3.4.3)

$$\pi_*(h \cdot f) = h \cdot \pi_*(f) \tag{3.4.4}$$

for $h \in N(H)$ and $f \in L^2(\mu \restriction \mathcal{B})$.

We show that

$$\mathcal{A} \subseteq L^2(\mu \restriction \mathcal{B}). \tag{3.4.5}$$

Indeed, since $L^2(\mu \restriction \mathcal{B})$ is a closed sublattice of $L^2(\mu)$, to see the above inclusion, it suffices to show that $k \cdot f_1 \in L^2(\mu \restriction \mathcal{B})$ for each $k \in N(H)$, for which it suffices to see that $k \cdot f_1$ is $\mathcal{B}$-measurable. The latter statement is a consequence of Claim 3.4.4.

Now it follows from (3.4.5), that $\pi_*(f)$ is defined for each $f \in \mathcal{A}$. Set

$$\mathcal{A}' = \{\pi_*(f)\colon f \in \mathcal{A}\} \subseteq L^2(\mu').$$

Since $\mathcal{A}$ is a closed sublattice of $L^2(\mu)$, $\mathcal{A}'$ is a closed sublattice of $L^2(\mu')$.

Claim 3.4.5. $\mathcal{A}' \supseteq L^\infty(\mu')$.

Proof of Claim 3.4.5. Since $\mathcal{A}'$ is a closed sublattice of $L^2(\mu')$, it is enough to show that for every $\varepsilon > 0$ and $B \in \mathcal{B}'$ there is $q \in \mathcal{A}'$ such that $\|\chi_B - q\|_2 < 2\varepsilon$. Let $\varepsilon > 0$ be given. Recall at this point the definitions of Q, $\mathcal{C}$, and π. It follows from these definitions that for $k \in Q$, $\pi_*(k \cdot f_1) \in L^2(\mu')$ has a continuous representative. For the remainder of the proof of this claim, we think of each $\pi_*(k \cdot f_1)$ as an actual continuous function from X' to $\mathbb{R}$. Let $L \subseteq X'$ be compact with $\mu'(L) > 1 - \varepsilon^2$. Let $C(L)$ stand for the lattice of all continuous functions from L to $\mathbb{R}$ with the L^∞-norm. By the definition of π, the sublattice of $C(L)$ generated by the constant functions and $\{\pi_*(k \cdot f_1) \restriction L\colon k \in Q\}$ separates points of L. By the Stone–Weierstrass theorem for lattices, see [2, Chapter 2, Theorem 11.3], this sublattice is dense in $C(L)$. This fact allows us to pick $q \in \mathcal{A}'$ with $\|(q \restriction L) - \chi_{B \cap L}\|_2 < \varepsilon$. Since $\mathcal{A}'$ is a lattice with $0, 1 \in \mathcal{A}'$, we can assume that $0 \leq q \leq 1$. Now, by the choice of L, we have $\|q - \chi_B\|_2 < 2\varepsilon$. □

Set

$$G_0 = N(H)/H.$$

We will need the claim below to define a Boolean action of G_0 on $\mathcal{B}(X')/\mu'$.

Claim 3.4.6. *For $g_1, g_2 \in N(H)$, if g_1, g_2 are in the same coset of H, then $g_1 \cdot [B']_{\mu'} = g_2 \cdot [B']_{\mu'}$ for each $B' \in \mathcal{B}(X')$.*

Proof of Claim 3.4.6. If $g_1, g_2 \in N(H)$ are in the same coset of H and $k \in N(H)$, then $g_1 k$ and $g_2 k$ are in the same coset of $k^{-1}Hk = H$. Thus,

$$((g_2k)^{-1}g_1k) \cdot f_1 = f_1,$$

hence

$$g_1 \cdot (k \cdot f_1) = g_2 \cdot (k \cdot f_1).$$

It follows from the definition of $\mathcal{B}$ that for each $B \in \mathcal{B}$ we have $g_1 \cdot [B]_\mu = g_2 \cdot [B]_\mu$. Now from this equality, by (3.4.3), we have for $B \in \mathcal{B}$

$$g_1 \cdot \pi_*([B]_\mu) = \pi_*(g_1 \cdot [B]_\mu) = \pi_*(g_2 \cdot [B]_\mu) = g_2 \cdot \pi_*([B]_\mu),$$

and the conclusion of the claim follows. □

Now Claim 3.4.6 allows us to define a measure preserving Boolean action of G_0 on $\mathcal{B}(X')/\mu'$ by letting for $gH \in G_0$ and $B \in \mathcal{B}(X')$

$$(gH) \cdot [B]_{\mu'} = g \cdot [B]_{\mu'}.$$

We consider the representation of G_0 induced by the above Boolean action. Since G_0 is locally compact, by a combination of the theorem of Mackey [29] and Theorem 3.4.2, G_0-continuous functions for the above representation are dense in $L^2(\mu')$. We will derive from it the conclusion that G-continuous functions are dense in $\mathcal{A}$ in the L^2-norm, which will finish the proof. First we note that if $f \in \mathcal{A}$ and $\pi_*(f)$ is G_0-continuous, then f is G-continuous. Indeed, obviously $\pi_*(f)$ is $N(H)$-continuous. By the fact that π_* preserves the L^∞-norm and by (3.4.4), it follows that f is $N(H)$-continuous. Since $N(H)$ is open in G, every $N(H)$-continuous function is also G-continuous; thus, f is G-continuous.

Now since $\pi_*(\mathcal{A}) = \mathcal{A}'$ and π_* preserves the L^2-norm, we will be done if we show that G_0-continuous functions are dense in $\mathcal{A}'$. But this follows from Claim 3.4.5 and the fact that G_0-continuous functions are dense in $L^2(\mu')$.

□

3.5 A property of the group $H_+([0,1])$

In this section we show that the condition $N(H)$ *is open* cannot be omitted in our characterization of groups of isometries of locally compact separable metric spaces. More precisely, we show the following proposition.

Proposition 3.5.1. *There is a Polish group which is* **not** *a group of isometries of a locally compact separable metric space, which satisfies the following weakening of the property* $(*)$. *For every open neighborhood* U *of the identity there is a closed subgroup* $H < G$ *such that* $H \subseteq U$ *and* G/H *is locally compact.*

Proof. We show that $G = H_+([0,1])$, the group of all increasing homeomorphisms of the interval [0,1] with the uniform convergence metric, is an example of such a group.

Take U, an open neighborhood of 1 in G. If necessary, we shrink U to

$$U_\epsilon = \{h \in G \colon \forall x |h(x) - x| < \epsilon\},$$

for some $\epsilon > 0$. Take

$$H_n = \{h \in G \colon h(\frac{k}{n}) = \frac{k}{n} \text{ for every } k = 0, 1, \ldots, n\},$$

a closed subgroup of G, such that $H_n \subseteq U_\epsilon$. Note that G/H_n is homeomorphic to

$$A_n = \{(a_1, a_2, \ldots, a_{n-1}) \colon 0 < a_1 < a_2 < \ldots < a_{n-1} < 1\}$$

(taken with the topology induced from the product topology) via the map

$$h \to \left(h(\frac{1}{n}), h(\frac{2}{n}), \ldots, h(\frac{n-1}{n}) \right).$$

Since A_n is an open subset of $[0,1]^{n-1}$, it is locally compact. Hence, G satisfies the weakening of the property $(*)$.

We show that G is *not* isomorphic to a Polish group of isometries of a locally compact separable metric space. For this, we show that the property $(*)$ is not fulfilled.

As shown in [10], there are exactly three proper normal subgroups of G. These are:

$$S_1 = \{h \in G \colon \mathrm{Fix}(h) \supseteq [0,p) \text{ for some } 0 < p < 1\},$$

$$S_2 = \{h \in G \colon \mathrm{Fix}(h) \supseteq (p,1] \text{ for some } 0 < p < 1\},$$

$$S_3 = S_1 \cap S_2,$$

where $\mathrm{Fix}(h)$ denotes the set of fixed points of h.

The group G is connected and non-locally compact. For the former, notice that G is a closure of the increasing union of

$$B_n = \{h \in G \colon h \restriction \left[\frac{k}{n}, \frac{k+1}{n}\right] \text{ is linear }, k = 0, 1, \ldots, n-1\},$$

and that B_n is homeomorphic to the connected space A_n.

Combining the remarks above, we see that for U, an open neighborhood of the identity which does not contain any of the S_1, S_2, S_3, there is no $H < G$, $H \subseteq U$ such that $N(H)$ is open. $\square$

Remark 3.5.2. The normalizer of H_n is equal to H_n.

Proof. Let $g \in N(H_n)$. Take $h_1 \in H_n$ such that $\mathrm{Fix}(h_1) = \{\frac{k}{n} \colon k = 0, 1, \ldots, n\}$. Take $h_2 \in H_n$ such that $gh_2 = h_1 g$. For each $k = 0, 1, \ldots, n$ we have $h_1(g(\frac{k}{n})) = g(\frac{k}{n})$. Since $g(\frac{0}{n}) < g(\frac{1}{n}) < \ldots < g(\frac{n}{n})$, and by the choice of h_1, we have $g(\frac{k}{n}) = \frac{k}{n}$, for each $k = 0, 1, \ldots, n$. Hence $g \in H_n$. $\square$

3.6 Non-turbulence for isometry groups

The notion of turbulence was introduced by Greg Hjorth. It is connected to the problem of classifiability of countable structures. Every continuous action by a locally compact group or by a permutation groups is non-turbulent. Recall that for a Polish group, a *Polish G-space* is a Polish space X together with a continuous action of G on X. Hjorth showed the following theorem.

Theorem 3.6.1 (Hjorth [20]). *Let $G = G_0 \times G_1 \times \ldots$, where each G_i is a permutation group or is locally compact. Then no Polish G-space is turbulent.*

This result was extended by Gao and Kechris.

Theorem 3.6.2 (Gao-Kechris [12]). *Let G be a Polish group of isometries of a locally compact separable metric space. Then no Polish G-space is turbulent.*

Below we show how to prove Theorem 3.6.2 using our characterization of Polish groups of isometries of locally compact separable metric spaces (Theorem 3.3.1). This proof is different from the proof due to Gao and Kechris. Our proof is a generalization of the proof of Theorem 3.6.1 due to Hjorth.

First we recall some definitions from [26]. Let X be a Polish G-space. For an open set U in X and an open symmetric $1 \in V$ in G define

$$R_{U,V} = \{(x, y) \in U \times U \colon \exists g \in V(g \cdot x = y)\}$$

For $x \in U$ we write $R_{U,V}(x) = \{y \in U \colon \exists g \in V(g \cdot x = y)\}$. Define the relation on U

$$x \sim_{U,V} y \iff \exists g_0, g_1, \ldots, g_k \in V(x_0 = x, x_{i+1} = g_i \cdot x_i, x_{k+1} = y, x_i \in U).$$

This is an equivalence relation. Let $\mathcal{O}(x, U, V)$ denote the equivalence class of x.

A point $x \in X$ is *turbulent* if for every U with $x \in U$ and every V, $\mathcal{O}(x, U, V)$ is somewhere dense, i.e. $\overline{\mathcal{O}(x, U, V)}$ has nonempty interior. The set of turbulent points is G-invariant. Therefore, we can talk about *turbulent orbits*. A Polish G-space X is called *turbulent* if every orbit is dense, meager, and turbulent.

Proposition 3.6.3 (Proposition 11.1 in [26]). *Let X be a Polish G-space with every orbit meager. Then for every U_0 there is $U \subseteq U_0$ and V such that $R_{U,V}$ is nowhere dense.*

We say that a Polish G-space X is *calm* if for every U, V and $x \in U$ there is $U' \subseteq U$ with $x \in U'$, and there is $V' \subseteq V$ such that $\mathcal{O}(x, U', V') \subseteq R_{U',V'}(x)$.

Proposition 3.6.4 (Propositions 11.3 and 11.5 in [26]). *(1) A calm G-space is not turbulent.*

(2) Any Polish G-space, where G is locally compact, is calm.

Let K be a Polish group. Let $d < 1$ be a left-invariant metric on K. Let X be a Polish K-space. Let H be a closed normal subgroup of K. Let d^* denote the quotient metric on $L = K/H$

$$d^*(k_1H, k_2H) = \inf\{d(k_1h_1, k_2h_2) \colon h_1, h_2 \in H\}.$$

Since H is normal in K and d is left invariant, d^* is also left invariant. Define

$$\mathcal{L}(L) = \{f \colon L \to [0,1] : \forall g, h \in L(|f(g) - f(h)| \leq d^*(g,h))\}$$

Therefore $\mathcal{L}(L)$ is just the space of 1-Lipschitz functions on L. We take $\mathcal{L}(L)$ with the pointwise convergence metric. We let L act on $\mathcal{L}(L)$ by left translations $(h \cdot f)(g) = f(h^{-1}g)$. We notice that $\mathcal{L}(L)$ is a compact Polish L space.

Let $F \colon X \to \mathcal{L}(L)^{\mathbb{N}}$, where $F(x) = (f^x_m)$ be given by

$$f^x_m(g) = d^*(g, \{h \colon \exists_{k \in K}(h = \pi(k) \text{ and } k \cdot x \in U_m)\}^{-1}), \tag{3.6.1}$$

where $(U_m)_m$ is a basis in X and $\pi \colon K \to L$ is the projection.

Recall that a function is *Baire class 1* if the preimage of each open set is F_σ. In the proof of Theorem 3.6.2 we will use a fact that Baire class 1 functions have a comeager set of continuity points.

Proposition 3.6.5. *Let K be a Polish group and let H be a closed normal subgroup of K. Let X be a K-space. Let L, $\mathcal{L}(L)$, and F be as above. Then:*

(1) for every $k \in K$ and $x \in X$, $F(k \cdot x) = \pi(k) \cdot F(x)$;

(2) $F\colon X \to F(X)$ is Baire class 1.

Proof. The proof of the first part is just a straightforward checking. For the second part, it is enough to show that for each m, $a \in \mathbb{R}$, $g \in G$, the set $\{x\colon f_m^x(g) < a\}$ is open (this is straightforward from the definition) and the set $\{x\colon f_m^x(g) > a\}$ is F_σ. For this we notice

$$f_m^x(g) > a \iff \exists_{b>a, b\in\mathbb{Q}}\{k \in K\colon d^*(\pi(k), g) < b\}^{-1} \cdot x \subseteq X \setminus U_m.$$

□

Proposition 3.6.5 is a generalization of a proposition due to Hjorth and Kechris, see [23], page 240 (see also [26], page 155). They show Proposition 3.6.5 assuming that K can be written as $L \times L_1$ for some closed subgroup L_1 of K.

Proof of Theorem 3.6.2. Let X be a Polish G-space. Let d be a left-invariant metric on G. Suppose, towards a contradiction, that this action is turbulent. Let $(O_n)_n$ be a decreasing sequence of neighborhoods of the identity in G. For every O_n take H_n and K_n, closed subgroups of G, such that $H_n \subseteq O_n$, $K_n = N(H_n)$ is open in G, and $L_n = K_n/H_n$ is locally compact. Let $\pi_n\colon K_n \to K_n/H_n$ be the projection. Denote by d_n^* the quotient metric on L_n (coming from the restriction of d to K_n). Let $F_n\colon X \to \mathcal{L}(K_n/H_n)^{\mathbb{N}}$ be defined as in (3.6.1).

Using the fact that for every n, the action of K_n/H_n on $\mathcal{L}(K_n/H_n)^{\mathbb{N}}$ is calm, and since we have for each n the equivariant and almost everywhere continuous map from the action of the open subgroup $K_n < G$ on X to the action of K_n/H_n on $\mathcal{L}(K_n/H_n)^{\mathbb{N}}$, we will deduce that the action of G on X is non-turbulent.

Fix an open set U in X and V_0, an open symmetric neighborhood of the identity in G, such that R_{U,V_0} is nowhere dense.

Claim. There are $n, x \in U$, and V_0'',V, open and symmetric neighborhoods of the identity, such that $V_0'' \subseteq V \subseteq V_0$, $H_n \subseteq V_0''$, $V = V_0''H_n$, $V \subseteq K_n$, $V^2 \cdot x \subseteq U$, F_n is continuous at x, and $R_{U,V^2}(x)$ is nowhere dense.

Proof of the Claim. Step 1: Find $\widetilde{U} \subseteq U$ and $V_0' \subseteq V_0$ such that $(V_0')^2 \cdot \widetilde{U} \subseteq U$ and $(V_0')^2 \subseteq V_0$.

Step 2: We find n, V, and V_0''. First, take V_0'' such that $(V_0'')^2 \subseteq V_0'$. Next, take n such that $H_n \subseteq V_0''$. Finally, let $V = V_0'' H_n$. Since $V \subseteq V_0'$, we have $V^2 \cdot \widetilde{U} \subseteq U$.

Step 3: We find x. Take $x \in \widetilde{U}$ such that F_n is continuous at x and $R_{U,V^2}(x)$ is nowhere dense. For the former we use $\{x \in X : F_n(x) \text{ is continuous}\}$ is comeager, for the latter we use $V^2 \subseteq V_0$ and we use the Kuratowski-Ulam theorem. For this x we have $V^2 \cdot x \subseteq U$. $\square$

We show that for appropriate $U' \subseteq U$, $x \in U'$, $V' \subseteq V$,

$$\overline{\mathcal{O}(x, U', V')} \subseteq \overline{R_{U,V^2}(x)}.$$

Since L_n is locally compact, the action of L_n on $\mathcal{L}(L_n)$ is calm. Take an open neighborhood U'' of $F_n(x)$ and take $V'' \subseteq \pi_n(V_0'')$, open and symmetric neighborhood of the identity, such that $\mathcal{O}(F_n(x), U'', V'') \subseteq R_{U'',V''}(F_n(x))$. Put $V' = \pi_n^{-1}(V'')$. Since $H_n \subseteq V_0''$ and $V = V_0'' H_n$, we have $V' \subseteq V$. Let $U' \subseteq U$ be an open neighborhood of x such that $F_n(U') \subseteq U''$. We show that these U' and V' work.

Let W be an open basic neighborhood with $W \cap \mathcal{O}(x, U', V') \neq \emptyset$. We show that $W \cap R_{U,V^2}(x) \neq \emptyset$. Pick $y \in W \cap \mathcal{O}(x, U', V')$. Then $F_n(y) \in \mathcal{O}(F_n(x), U'', V'')$. Therefore, $F_n(y) \in R_{U'',V''}(F_n(x))$, i.e, for some $g' \in V'$,

$$F_n(y) = \pi_n(g') \cdot F_n(x) = F_n(g' \cdot x).$$

Let m be such that $W = U_m$. We have, in particular, $f_m^y = f_m^{g' \cdot x}$, where we write $(f_m^x) = F_n(x)$. Since $y \in U_m$, we have $f_m^y(1_{L_n}) = 0$, and therefore $f_m^{g' \cdot x}(1_{L_n}) = 0$. It follows that there is a sequence $h_i \in L_n$ with $h_i \to 1_{L_n}$, and there is a sequence $k_i \in K_n$ with $\pi_n(k_i) = h_i$ and $k_i \cdot g' \cdot x \in U_m$. For large $i, h_i \in V''$, so $k_i \in V'$, and since $g' \in V'$, we have $k_i \cdot g' \cdot x \in (V')^2 \cdot x \subseteq U$. Hence $W \cap R_{U,V^2}(x) \neq \emptyset$.

Therefore $\overline{\mathcal{O}(x, U', V')} \subseteq \overline{R_{U,V^2}(x)}$. Since $\overline{R_{U,V^2}(x)}$ is nowhere dense, we get a contradiction. $\square$

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